STAIR CASES

STRUCTURAL ANALYSIS AND DESIGN

M.Y.H. BANGASH
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Contents

Preface

In recent years free-standing and geometric (spiral, helical, elliptical and combinations) staircases have become quite popular. Many variations of these staircases exist. A number of researchers have come forward with different concepts in the fields of analytical, numerical, design and of experimental assessments. The aim of this book is to cover all these methods and to present them with greater simplicity to a practising engineer. The numerous examples which are given in the text will obviously make that task easier. The book is divided into five chapters. Chapter 1 deals with the general requirements for analysing, designing and structural detailing of staircases in various materials. This chapter will assist with the analysis and design of staircases given in Chapters 2 and 3. Chapter 2 is devoted to all available classical methods including those developed by Taleb, Gould, Liebenberg, Siev, Morgan and Cohen. Examples of staircases using these methods are included. This is followed by Chapter 3, which is devoted to staircases analysed by the flexibility, the stiffness and the finite element methods. A comprehensive treatment of staircases is given, analysed by plate/shell membrane technique. All methods mentioned in Chapter 3 are fully described and reasonably supported by numerical examples. Analyses stated in Chapters 2 and 3 are relevant to all materials. Chapter 4 is earmarked for a comparative study of some of the methods described earlier. Charts and graphs are given for the reader to examine for himself or herself the capabilities of all these methods and their relevant applications. Numerous design examples are given on free-standing and geometric staircases and their elements and components. The design examples are related to the case studies given in earlier chapters, which are based on existing staircases.

Bibliographical references have been given in the text for those who wish to carry out in-depth studies in one or all areas of research. The book is supported by two appendices for additional analyses and examples of staircases from the practices of different countries.

Appendices will particularly be of interest to those practising engineers who wish to make a comparative study of practices and code requirement of various countries.

The book will serve as a useful text for teachers preparing design syllabuses for undergraduate and post graduate courses. Each major section has been fully explained to permit the book to be used by practising engineers and students, particularly those facing the formidable task of having to design/detail complicated staircases with unusual boundary conditions for specific contracts and research assignments. Contractors will also find this book useful in the preparation of construction drawings.

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The authors are grateful to Shyam Shrestha for typing the entire manuscript and to Marlyn Grazzette for preliminary work on typing certain passages.

Definitions

Baluster:	An infilling member of a balustrade.
Clearance:	The unobstructed height measured at right angles to the pitch line.
Depth of tread:	The horizontal distance to the face of the riser.
Elliptical stair:	A flight described on plan as an ellipse.
Going:	A horizontal part of a step.
Helical stair:	A stair rising to describe a helix and in all the treads are tapered on a plan. (Commonly known as spiral or circular stair.)
Landing:	A horizontal platform of the flight at the end or between flights.
Half landing:	A landing at which a half turn is made between two flights of stairs.
Scrolls:	The end of a handrail sculptured to resemble a roll of parchment.
Strings/stringers:	Beams which support the stair flights.
Tread (parallel):	A step at which the nosing is parallel.

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Major notations

A	= constant;
B	= width of stairway;
b	= width of the supporting beam;
D, D_f	= stair thickness;
d	= effective depth;
E	= Young's modulus;
f'_c, f_{cu}	= concrete strength, cylindrical and cubic, respectively;
G	= shear modulus;
$\overline{G}$	= going;
H_o, H	= horizontal redundant force;
I	= second moment of area;
J	= polar moment of inertia; Jacobian;
K, K_H	= spring constants;
L	= spans plane projection etc.;
M_v, M_o, M	= bending moments in specific location;
M_t, T	= twisting moment;
M_{nf}, M_{rf}	= lateral moment (about axis normal to the stair and vertical moment about the horizontal axis), respectively;
P_{nf}	= axis force;
r, R_1	= radius of centre line of load;
r_i, R_2	= radius of centre line of steps;
V_{hf}, V_{nf}	= radial horizontal shear force and shear force across the section of stairs, respectively;
T_f	= torsional moment;
T_o	= limiting value for maximum torsional moment;
P	= horizontal component of membrane force load;
Q	= membrane force;
q_L	= load on the flight;
U	= strain energy;
W	= total design ultimate load;
w, q, p	= uniform load;
x, y	= shorter and longer overall dimensions of rectangular cross section, respectively;

x, y, z	=	coordinates;
β	=	total arc subtended by helix or angle of inclination;
γ_1	=	particular value of theta at which torsional moment is maximum or particular value of theta at which the vertical moment is equal to zero (inflexion point);
λ_1, λ_2	=	parameters;
Θ	=	angle subtended in plan measured from midpoint of stair;
ϕ	=	slope made by tangent to helix centre line with respect to the horizontal plane;
x	=	parameter;
ε	=	strain;
τ	=	stress, shear stress;
τ_r	=	radius (specific).

Comparison of notations

Text	European	American
B	b	$b,\ w_f$
L_1, L_2	a	$l,\ a$
L	l	L
$\overline{G}$	$g,\ G$	T
h_1	s	R
$D,\ D_f$	t	$t,\ d$
w_d, w_L	$g,\ q$	P_d, P_L, W_D
ω, δ, Δ	Δ, y	δ, Δ
M_t, T	M_t	M_T

For other symbols refer to individual chapters.

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Conversion factors (units)

Imperial or MKS units	SI units
1 inch	= 25.4 mm;
1 foot	= 0.3048 m;
1 ft/s	= 0.3048 m/s;
1 ft^2	= 0.0929 m^2;
1 in^2	= 645.2 mm^2;
1 ft^3	= 0.02832 m^3;
1 rad	= 57.296 deg;
1 lb	= 0.454 kg;
1 ton (short) = 2000 lb	= 0.9072 Megagram (Mg);
1 ton	= 9.964 kN;
1 lbf	= 4.448 N;
1 kip	= 4.448 kN;
1 kip/ft	= 14.594 kN/m;
1 kip/in	= 175.1268 kN/m;
1 kgf	= 9.806 N;
F° (Fahrenheit) to °C;	$t_c = (t_f - 32)/1.8$ or $t_c = 5/9$ Kelvin;
°C (Celsius) to F°;	$t_f = 1.8t_c + 32$;
1 lb/ft^3	= 16.018 kg/m^3;
1 cu ft	= 16.4 cm^3;
1 cu yd	= 0.765 m^3;
1 lb/in^2	= 6.89 kPa (kN/m^2);
1 lb/ft^2	= 47.880 Pa (N/m^2);
1 lbf/ft	= 14.59 N/m;
1 kip/in^2	= 6.895 MPa (MN/m^2).

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CHAPTER 1

Specifications and basic data on staircases

1.1 INTRODUCTION TO STAIRCASES

A stair is constructed with steps rising without a break from floor to floor, or with steps rising to a landing between floors, with a series of steps rising further from the landing to the floor above. There are three basic ways in which stairs are planned:

A *straight flight stair* (Fig. 1.1), which rises from floor to floor in one direction with or without landing.

A *quarter turn stair* (Fig. 1.2), which rises to a landing between floors, turns through 90°, then to the floor above.

A *half turn stair* (Fig. 1.3), which rises to a landing between floors, turns through 180°, then rises, parallel to the lower flight, to the floor above. This type of stair is sometimes called 'dog-leg' or 'scissor-type stair'.

Geometric stairways. The stairs mentioned above are generally free-standing ones. In addition to these, stairs known as geometrical stairs can be designed into spiral, helical, circular, elliptical (Fig. 1.4) and other shapes. They can all be in concrete, steel, timber or combination. The stairs are sometimes described as open well stairs where a space or well exists between flights (Fig. 1.2(c)).

Again in free-standing stairs the main types are:

– Type 1: Those supported transversely or across the flight. Stringer beams are needed (Fig. 1.1) on one or both sides.

– Type 2: Those spanning longitudinally along the flight of steps (Fig. 1.2) either on walls or on landing beams or on wall beams.

– Type 3: Cantilever type projecting from walls or wall beams (Fig. 1.5) with each step acting as a cantilever.

– Type 4: Combination of Type 2 and Type 3. Every 4th or 5th step is cantilevered with sloped soffit with a slab continuous between two steps.

The structural details of some of the stairs are given in Appendix 2.

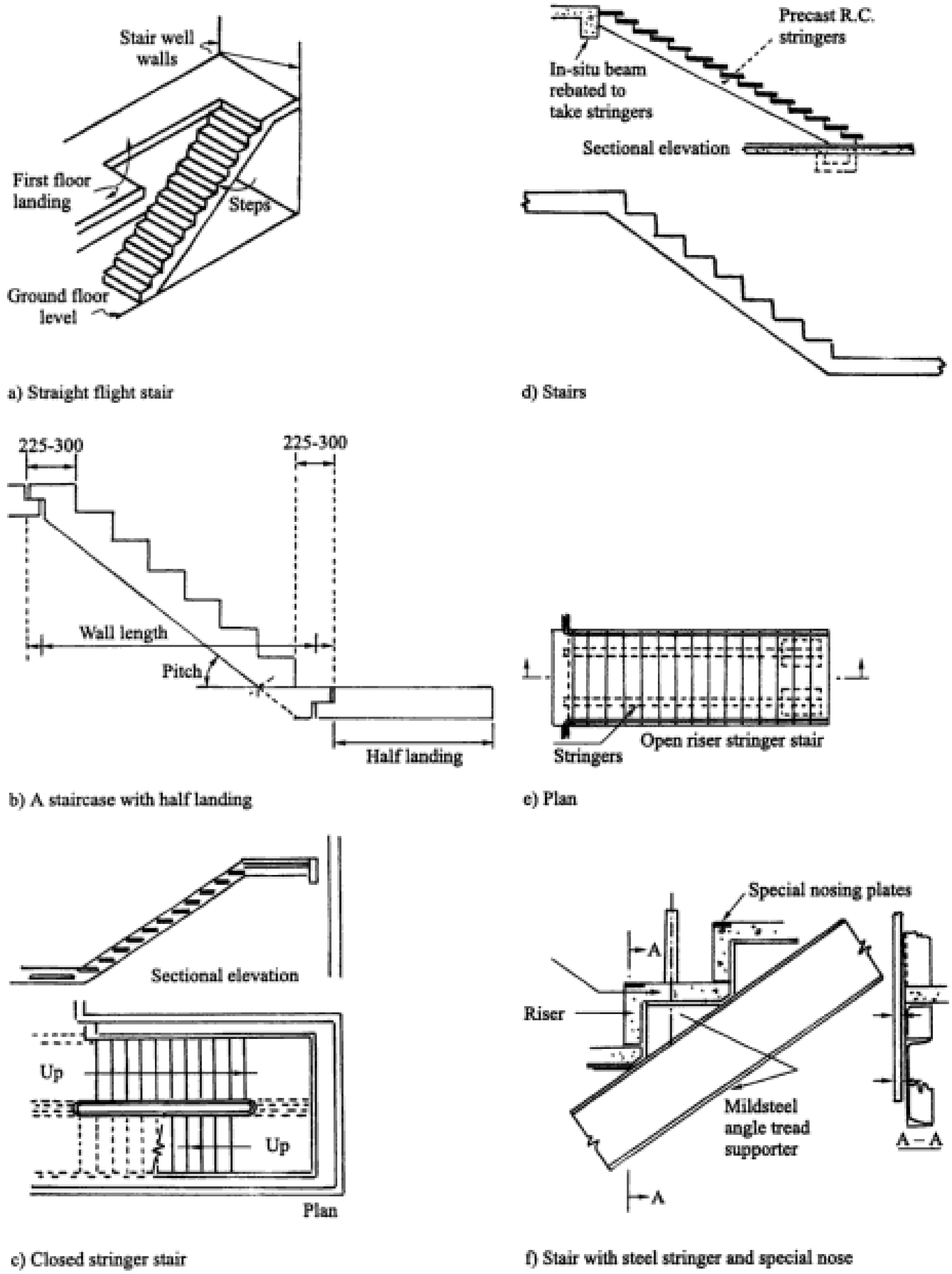
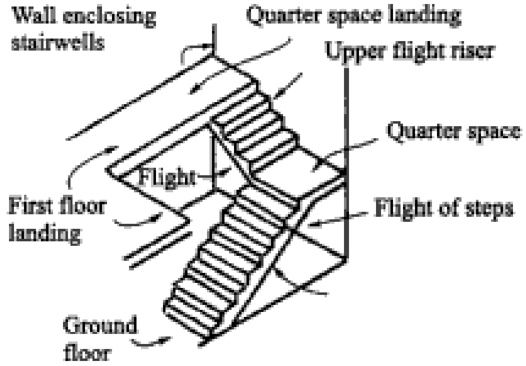
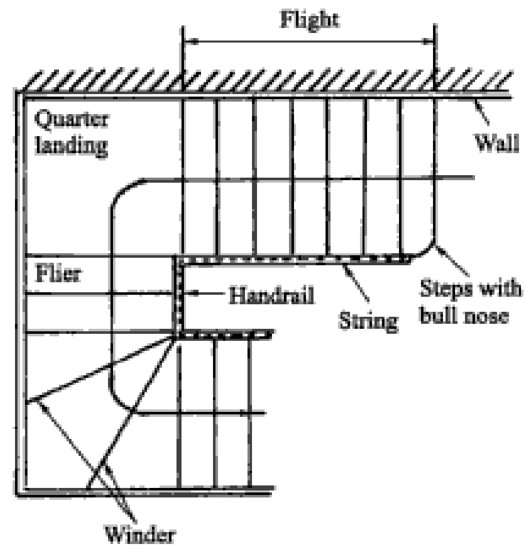


Figure 1.1. Straight flight stairs.



a) Elevation quarter turn stair



b) Plan with a quarter turn

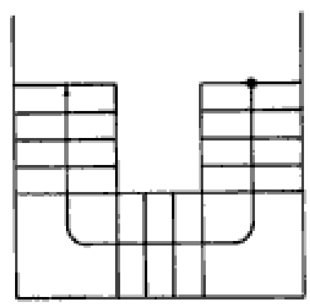


Figure 1.2. Quarter turn stairs.

c) Open-well stair, quarter-turn type with a horizontal space.

1.2 STAIRWAY LAYOUTS

Stairway layouts depend on several factors including building type and its layout, choices, material etc. Comfortable stairways should be designed in relation to the dimensions of the human figure. A summary of the American practice for staircases is given in Tables 1.1 and 1.2. The British Standard on stairs BS5395 (1977) defines some of these dimensions in Figure 1.6. The British and the European practices use the following criteria for width, length and headroom etc.:

a) Flats – two storey to four storey $w_F = 900$ mm; more than four storey $w_F = 1000$ mm.

b) Public buildings using each floor – under 200 persons $w_F = 1$ m; 200 to 400 persons $w_F = 1.5$ m; in excess of 400 persons 150 mm to $w_F > 3$ m. Where the width is 1.8 m or over, the width should be divided by a handrail.

c) The length and rise a minimum of 3 steps and a maximum of 16 steps. There must not be more than 36 rises in consecutive flights

Plate 1.1. Open riser stringer stair straight flight.

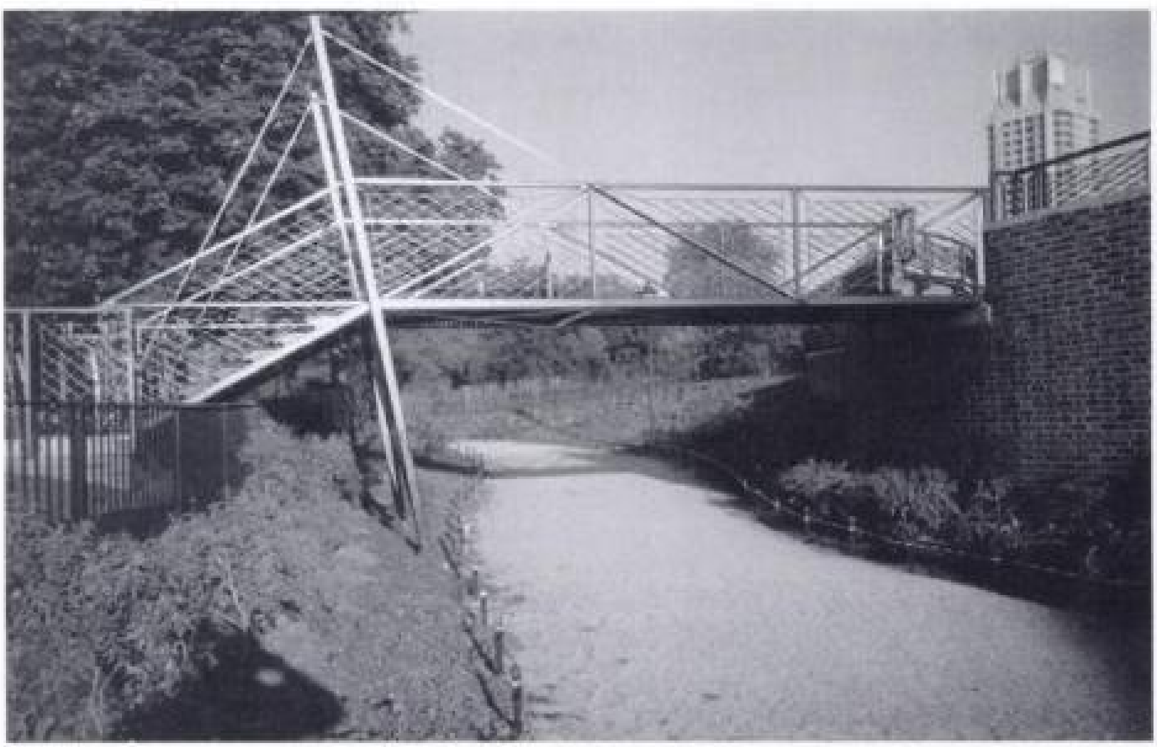
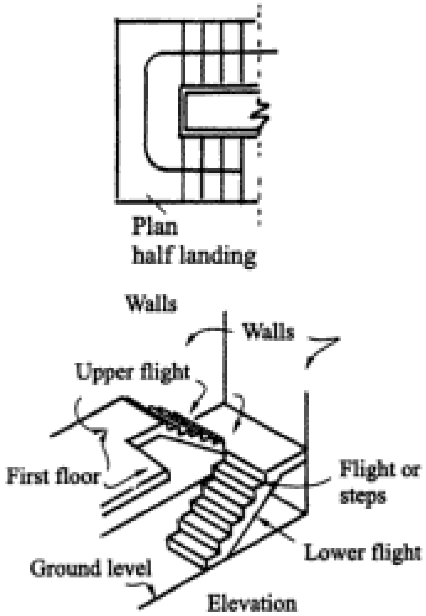
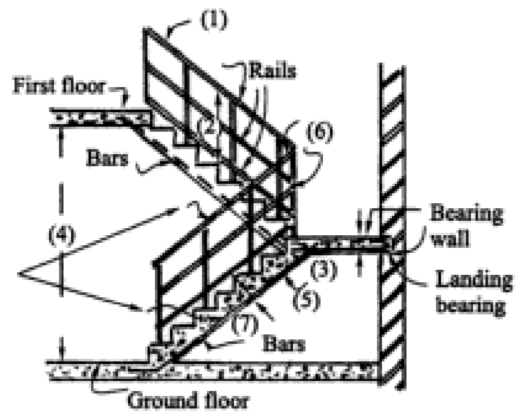


Plate 1.2. Straight stairs and long landing using cable stays.

Plate 1.3. Close stringer stairs (quarter turn). (With compliments from the Institution of Civil Engineers, London.)

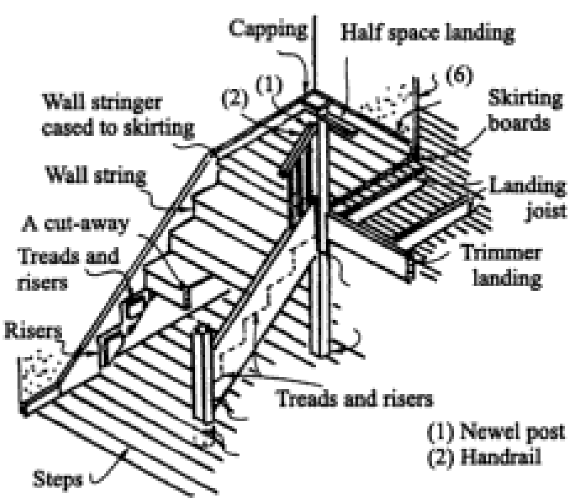


a) Dog-leg stair

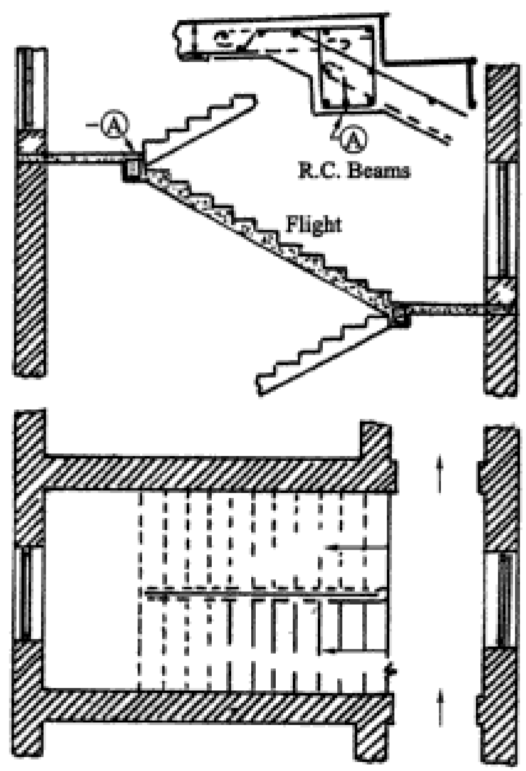


(1) Balustrade
(2) Clear height
(3) Landing depth
(4) Effective height
(5) Cover
(6) Post
(7) Waist

c) Lower flight of a half-turn stair



b) Staircase in timber-lower flight



d) Scissor type-landing slabs
on beams or stringers

Figure 1.3. Half turn stairs.

Plate 1.4. Scissor type stairs.

without a change in the direction of travel of 30° or more. The total rise must not exceed 6 m.

1.2.1 *Landings, landing beams and flights*

A quarter space landing in wood is generally supported by a newel post carried down to the floor below. In small houses quarter or half turn stairs are sometimes constructed with winders (Fig. 1.2(b)) instead of quarter or half space landings. Winders are triangular shaped steps constructed at the turn from one flight to the next. The landing beams (Fig. 1.9) are designed as rectangular or flanged beams, for the reactions from the two flights or steps on one side and the landing on the other.

Plate 1.5. Helical stair case in steel.

1.2.2 *Strings or stringers*

These are available in steel, concrete, timber and composite. There are two types of wood string, namely, the open (cut) and the close (closed) strings. The string enclosed treads and risers are shown in (Fig. 1.1). In wood their top edges project some 50 to 60 mm above the line of nosing or tread. Wall strings are closed ones. The outer strings, particularly those made in timber, are cut to the profile of the treads and risers and are secured by wood bearers screwed to both strings and treads or risers in the underside of the flight. A cut out string is shown in Figure 1.10.

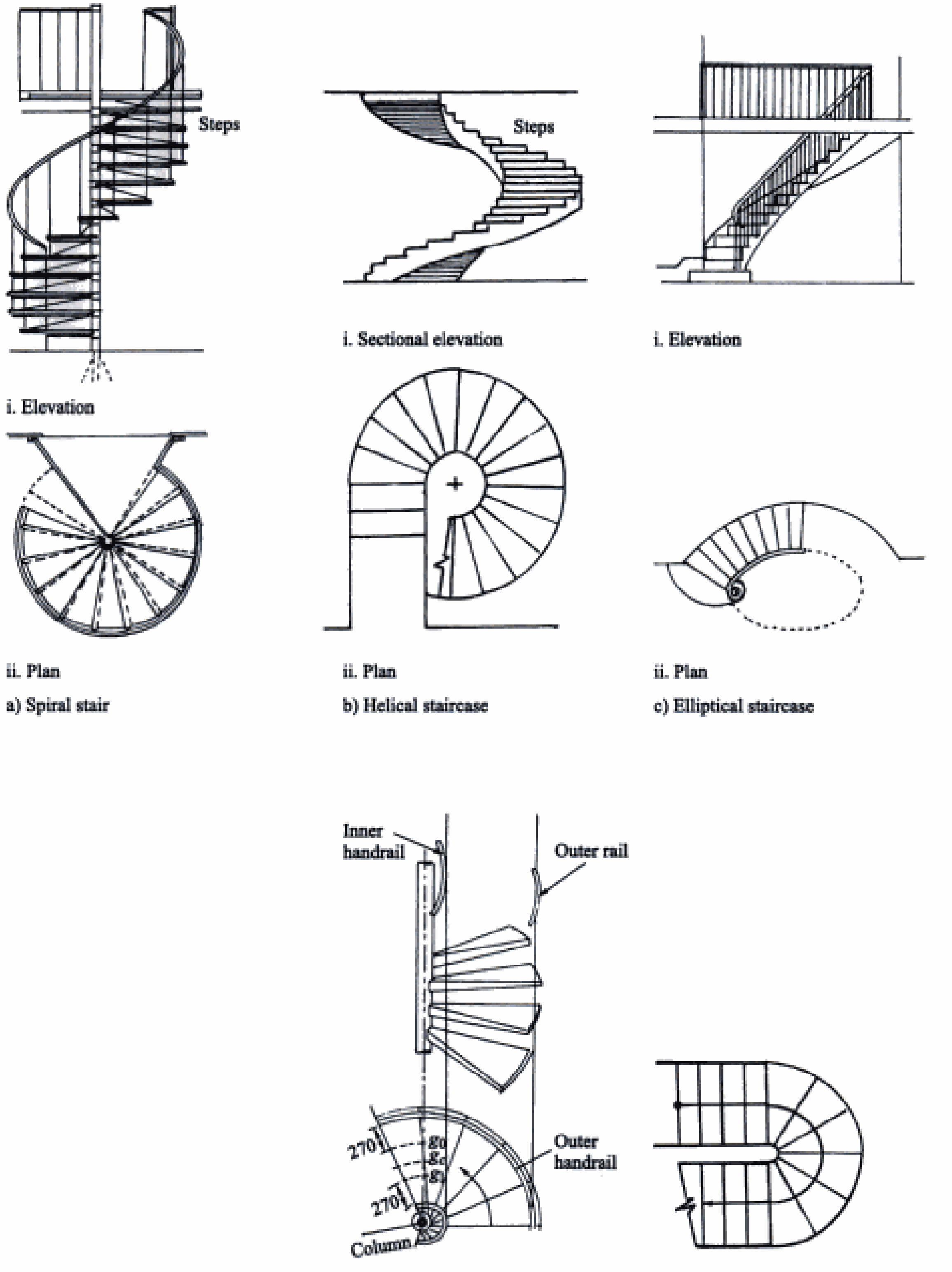
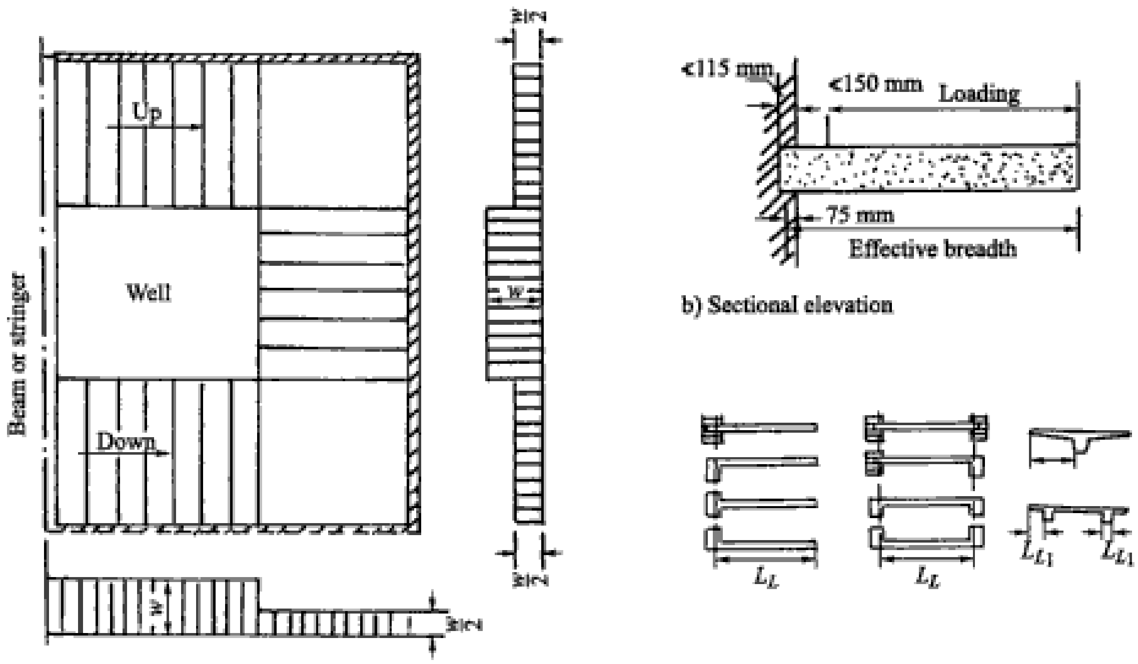


Figure 1.4. Geometrical stairs.

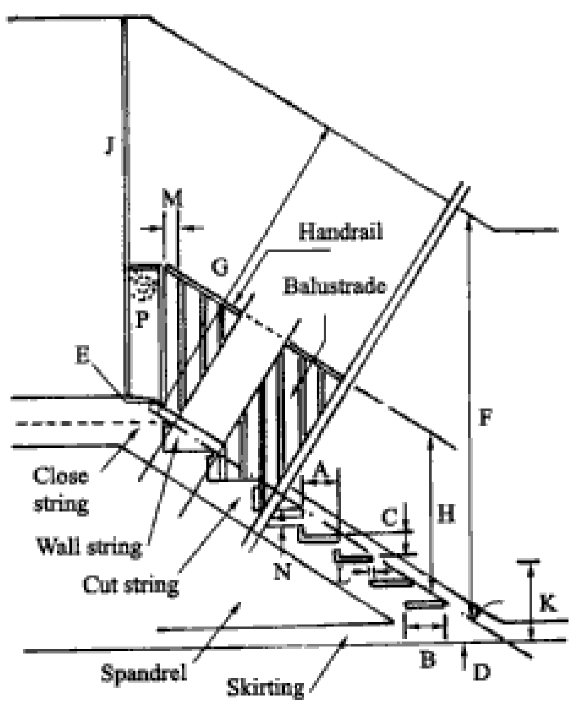


a) Plan

b) Sectional elevation

c) Various boundary conditions

Figure 1.5. Free-standing stair-cantilever type.



Key

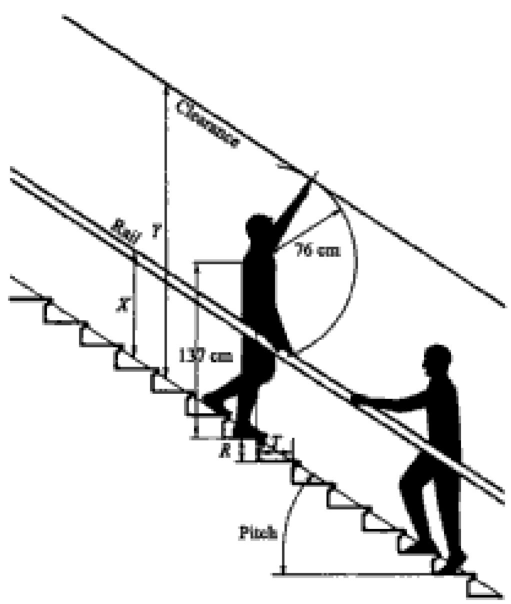
A going = G
B tread depth
C rise = h_i
D pitch
E pitch line
F headroom
G clearance
H balustrade height
J storey post
K handrail required when rises more than 600 mm
L overlap
M gaps in balustrades
N openings between adjacent treads
P scroll or returned end

Figure 1.6. Definitions based on British Standard (BS5395, with compliments from BSI).

Table 1.1. Dimensions for stairways.

Step dimensions		Gradient designations		Headroom	Handrail height
Riser $R = h_i$ (cm)	Tread $T = \overline{G}$ (cm)	Per cent grade	Angle in deg-min	Y (cm)	X (cm)
12.70	40.64	31.25	17-21	215.9	85.09
13.335	39.37	33.87	18-43	218.4	85.09
13.97	37.465	37.28	20-27	218.4	85.09
14.605	35.56	41.07	22-20	218.4	85.09
15.24	34.29	44.44	23-58	220.9	83.82
15.875	33.02	48.07	25-40	220.9	83.82
16.51	34.15	53.06	27-57	223.5	83.82
17.145	29.845	57.44	29-52	223.5	83.82
17.78	27.94	63.63	32-28	226.0	83.82
18.415	26.67	69.04	34-37	228.6	83.82
19.05	25.40	75.00	36-52	231.4	83.82
19.685	24.13	81.57	39-12	236.2	85.09
20.32	22.86	88.88	41-38	238.7	85.09
20.955	21.59	97.05	44-9	243.8	85.09
21.59	20.955	103.02	45-51	248.8	85.09
22.225	20.6375	107.07	46-57	263.4	85.36
22.86	20.32	112.50	48-22	251.5	86.36

Minimum for head clearance only can be safely taken as 213.36 cm for all gradients; HUD permits 203.2 cm.
Notes: 1. Consult local building codes on all stair problems. 2. All steps are laid out by the proportion 17.78 cm × 27.94 cm. 3. Risers from 16.83 cm to 19.37 cm are most comfortable for interior stairs. 4. The minimum width for single file travel is 76 cm but 91.5 cm is more comfortable. A width desirable for furniture passage shall be 107 cm.

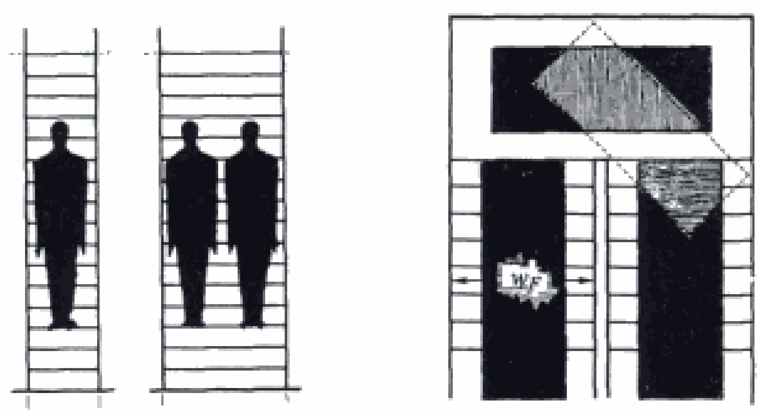


1.2.3 *Balustrade*

The wood balustrade is vividly described in Figure 1.11. Since a metal balustrade is commonly used with concrete stairs, it is difficult owing to numerous techniques, to give the meaning full details. One typical illustration is given in Figure 1.12.

Table 1.2. Recommended minimum clear widths of stairs for furniture movement (cm) w_F.

Furniture		Minimum headroom			Unlimited headroom	
Article	Size (cm)	Wide U type	Narrow U type	Wide and narrow U type	Narrow only U type stair	Narrow only landing
Double bed box spring	137×198×20.3	96.5	111.8	68.6	–	–
Dressing table	56 × 122 × 76	73.7	73.7	73.7	–	–
Divan-club	106 × 218 × 84	142.2	142.2	101.6	91.4	111.8
Divan-average	91 × 203 × 76	132	132	88.9	–	–
Piano-concert grand	274 × 163 × 60	142.2	142.2	96.5	91.4	101.6
Piano-draving room grand	221 × 157 × 46	116.8	116.8	91.4	–	–
Sideboard	53 × 152 × 97	76.2	76.2	76.2	–	–
Buffet	89 × 99 × 193	121.9	121.9	86.4	–	–
Dresser	53 × 183 × 162	132	106.7	101.6	91.4	11.8
Table (6 people)	106 × 152 × 76	96.5	96.5	96.5	91.4	101.6
Table (8 people)	106 × 213 × 76	142.2	132.1	96.5	91.4	101.6
Table (10 people)	193∅	142.2	142.2	91.4	–	–
Desk-slop top	76 × 122 × 99	99.1	96.5	96.5	91.4	101.6
Desk-flat top	91 × 183 × 76	99.1	96.5	96.5	–	–
Desk-executive's	96 × 183 × 76	127	127	94	91.4	96.5
Trunk-wardrobe	58 × 76 × 109	73.7	73.7	73.7	–	–



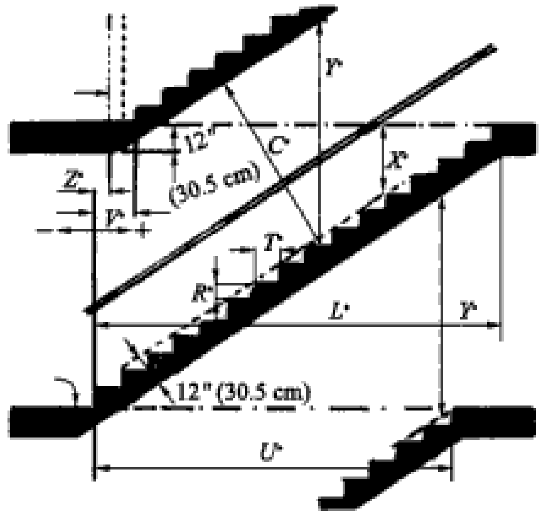
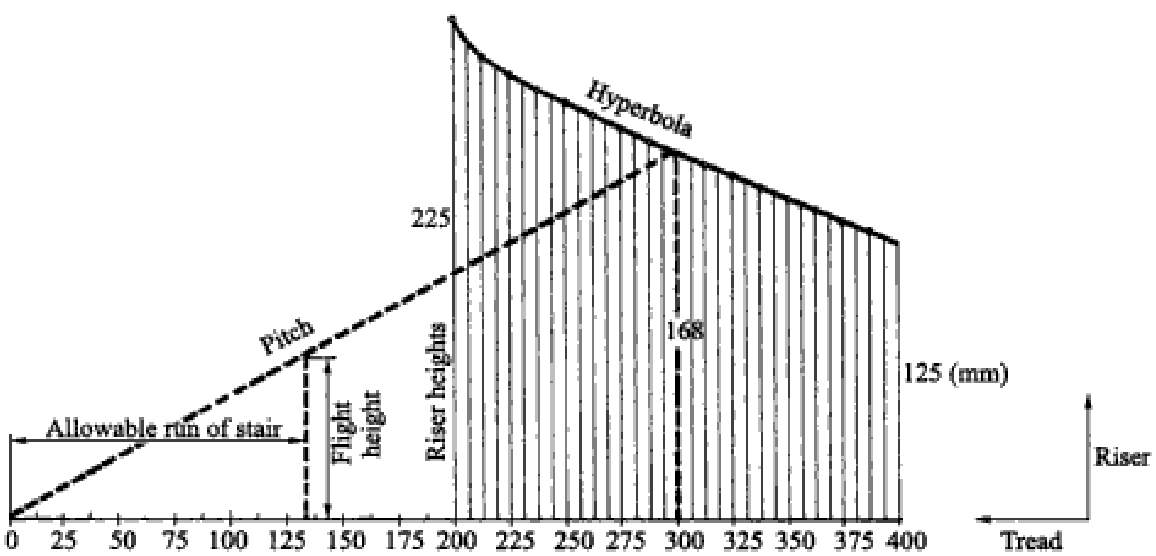


Figure 1.7. A tread-riser diagram (Time-saver's Standards 1991). Dimensions are accurate to half-full size, thus, reading can be made directly without need for calculation.

Notes:

1. Steps to find the proper riser for a given tread. Read tread line to a given width and select riser at intersection.
2. To find proper tread for a given riser: Select riser to nearest 3 mm and read tread width to nearest 13 mm (or nearest 6 mm by interpolation) at intersection with tread line.
3. To find tread and riser for given height and run of stair: scale run of stair on tread line. Draw flight to flight height at same scale. Draw pitch of stair. Where pitch intersects hyperbola, measure riser (at half-full size) to tread line. Read tread width directly or measure at half-full size.
4. To find run of stair for given height, tread and riser: select riser. Connect intersection at hyperbola with 0 on tread line, thus, establishing a pitch. Draw flight to flight height to scale, intersecting pitch and perpendicular to tread line. Run is found at same scale as height on tread line from 0 to intersection of flight to flight height.

Figure 1.8. Relationship between rise and going (with compliments of the British Standards BS5395, 1977).



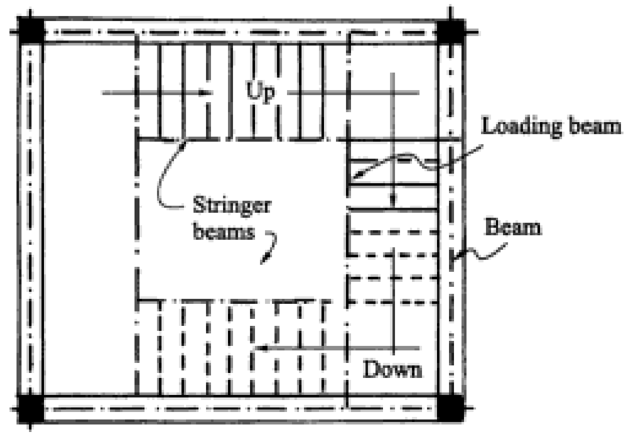


Figure 1.9. Three flights stairs-position of landing beams.

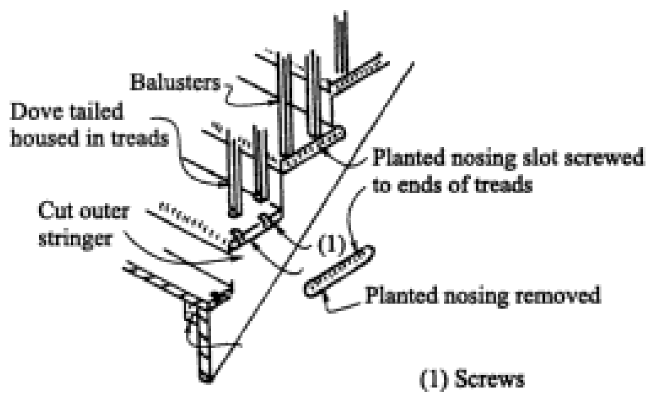


Figure 1.10. A cut out string with balusters.

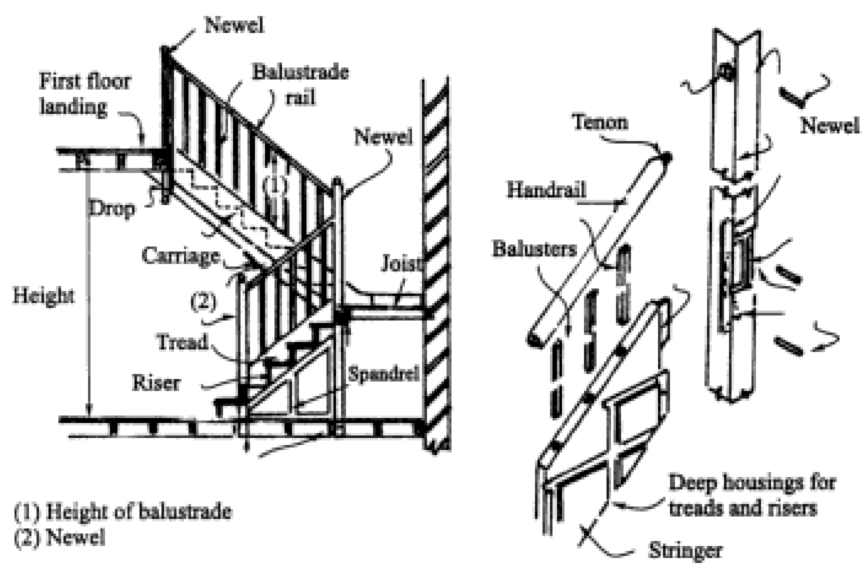
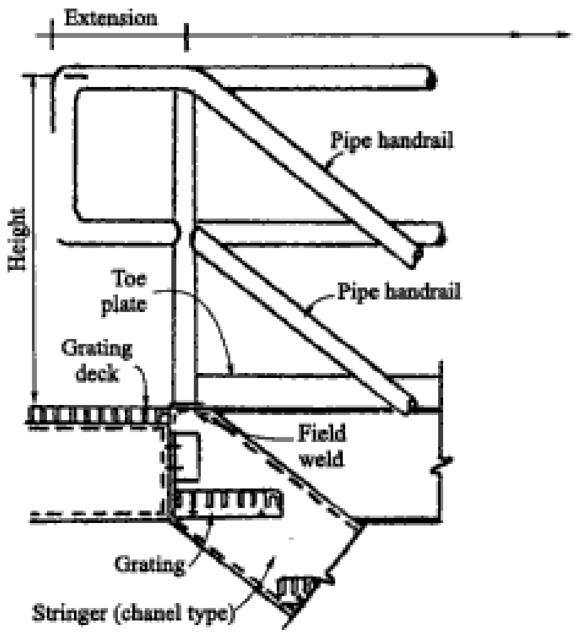
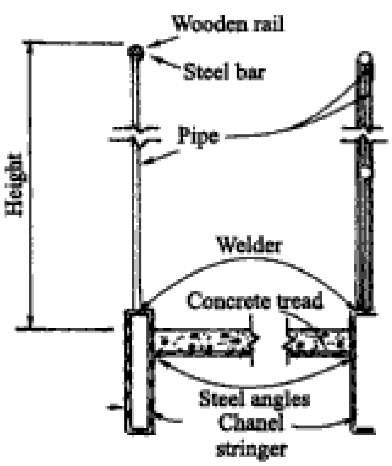


Figure 1.11. Wooden balustrades.



a) Balustrade

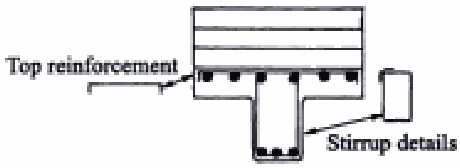


b) Stringer and handrail sections

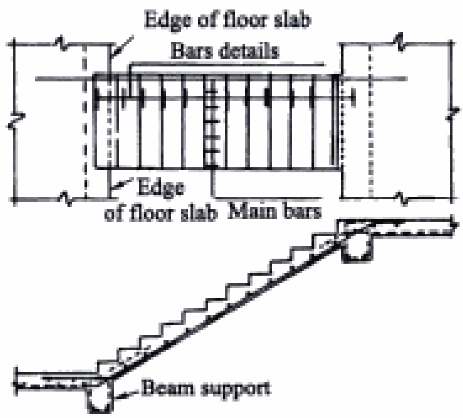
Figure 1.12. Steel balustrade.

1.2.4 *Free-standing stairs and their reinforcement layouts*

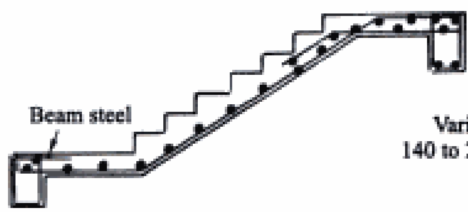
Some of the free standing stairs have been described in Section 1.2. This section is devoted entirely to this type of stairs in reinforced concrete. The support arrangement influences the reinforcement layout. Figure 1.13(a) shows the cross section detail of a staircase supported on a central beam. Since each side of the staircase is acting as a cantilever, the main reinforcement is placed on top with distribution steel. The entire cross section looks like a T-beam. Figures 1.13(a) and 1.13(b) give reinforcement for the sectional elevation and the plan of a straight stair-flight supported at each end by cross beams lying between the flight and the landings. Figure 1.13(c) shows the reinforcement detail when the cross beams are placed at the end of the landing. When the top landing is supported by the brick-wall, Figure 1.13(c) is modified and this is shown in Figure 1.13(d). When the flight is supported on side beams, the reinforcement details are shown in Figure 1.13(e). It is essential to mention various types of concrete steps, namely the cast-in-situ and precast concrete steps. They are self evident in Figures 1.13(f) and 1.13(g). Straight stair-flights and landings supported by side or centre beams as shown in Figure 1.13(h) will require cranked beams. The structural details depend on the ratio between horizontal sections and the sloping section. Figure 1.14 shows the stringer beams reinforcement layout for a two-flight staircase with landing. It is interesting to note the reinforcement layout for the downstand part of the stringer beams.



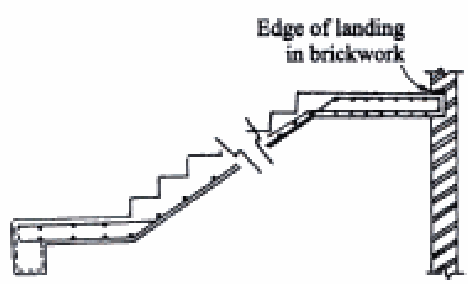
a) Staircase on central beam



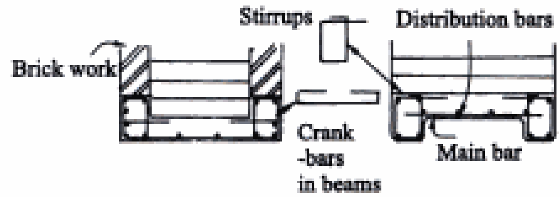
b) Straight stair flight-continuous over landing beams



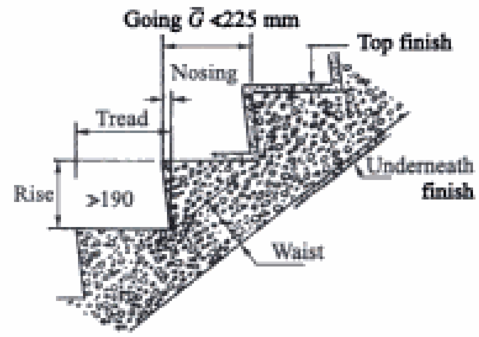
c) Straight stair flight-landings with cross beams at ends



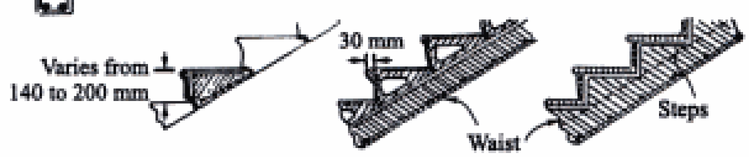
d) Straight stair flight on brickwork



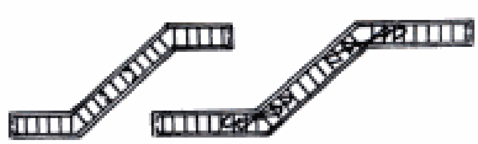
e) Staircase on side beams



f) Cast-in-situ concrete steps with protective finish



g) Precast concrete steps



h) Cranked beams

Figure 1.13. Reinforcement for free-standing stairs.

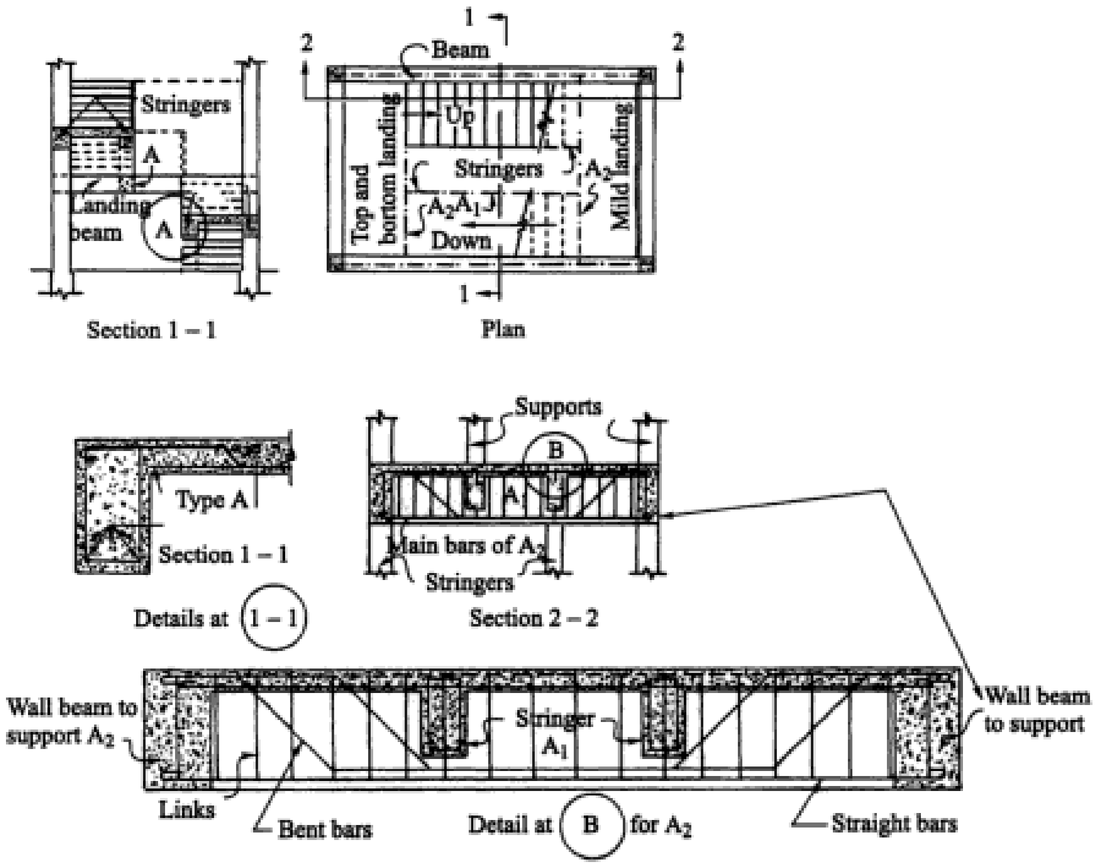


Figure 1.14. Layout and reinforcement details of stringer beams in relation to the main reinforcement of stairs.

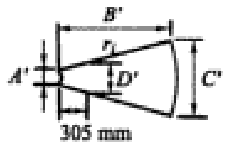
1.2.5 *Data for geometric stairways*

A brief introduction to these staircases is given in Section 1.1. It is vital to give brief data on spiral/helical staircases. These staircases are manufactured in a variety of diameters. The most common materials for tread and platform are steel, aluminium and wood. Steel and aluminium can be smooth plate, checker plate, pan or tray type and bar. A variety of hardwoods can be used. For exterior or wet area interiors zinc-chromated rust inhibitor, black acrylic enamel and black epoxy are usual. Platform dimensions usually are 2″ (50 mm) larger than the stair radius. Table 1.3 gives specifications for spiral and helical stairs. Where horse-shoe shapes

Table 1.3. Specifications for spiral/helical stairs parameters.

Diameter (cm)	101.1	121.9	132.0	152.4	162.6	182.9	193.0	223.5	243.8
Centre column (cm)	10.1	10.1	10.1	10.1	10.1	10.1	10.1	16.8	16.8
Weight (kg)	93.9	99.8	106.6	113.4	120.2	140.6	147.4	197.3	220.0
Tread detail A' (cm)	10.1	10.1	10.1	10.1	10.1	10.1	10.1	16.8	16.8
Tread detail B' (cm)	45.7	55.9	61.0	71.1	81.3	86.4	91.4	106.7	121.9
27 tread detail C' (cm)	23.5	28.3	30.8	32.7	37.8	42.5	44.8	52.0	56.7
27 tread detail D' (cm)	19.4	20.3	21.0	22.2	21.6	21.9	21.0	25.4	26.7
30 tread detail C' (cm)	26.7	31.9	34.6	39.7	42.5	48.0	50.5	58.4	63.8
30 tread detail D' (cm)	21.6	21.9	22.2	22.5	22.9	23.5	23.8	28.9	29.2
Landing size (cm)	55.9	66.0	71.1	81.3	86.4	96.5	101.6	116.8	132.0

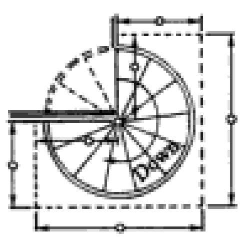
Table 1.3 (cont.).



a) Sector in plan

b) Circular in plan

c) Horse-shoe in plan

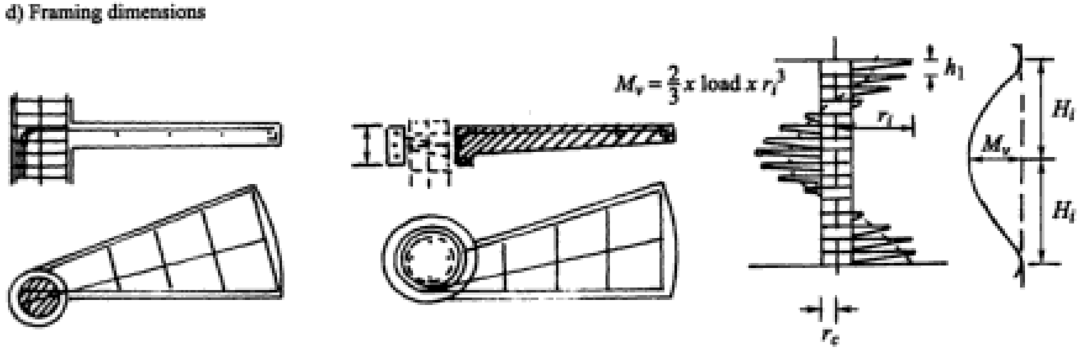


d) Framing dimensions

Framing dimensions (cm)

Stair diameter	2.54	5.08	7.62	10.16	12.70
40	50.8	50.8	67.0	111.8	111.8
48	67.0	67.0	71.1	132.0	132.0
52	66.0	66.0	76.2	142.2	142.2
60	76.2	76.2	68.4	162.6	162.6
64	81.3	81.3	91.4	172.7	172.7
72	91.4	91.4	101.6	193.0	193.0
76	96.5	96.5	106.7	203.2	203.2
88	111.8	111.8	121.9	233.7	233.7
96	121.9	121.9	132.0	254.0	254.0

27 riser table			30 riser table	
Finish floor height (cm)	Number of steps	Circle degree	Finish floor height (cm)	Number of steps
228.6-243.8	11	297	215.9-241.3	9
246.4-264.2	12	324	243.8-264.2	10
266.7-284.5	13	351	266.7-289.6	11
287.0-304.8	14	375	292.1-312.4	12
307.3-325.1	15	405	315.0-337.8	13
327.7-345.4	16	432	340.4-360.7	14
348.0-365.8	17	459	363.2-386.1	15
368.3-386.0	18	486	388.6-408.9	16
388.6-406.4	19	513	411.5-434.3	17
408.9-426.7	20	540	436.9-457.2	18



e) Reinforcement details of treads and columns

Formulae

$\overline{G}$ (outer going) $= 2(r_i - 270°)\sin\theta/2$

Clear headroom $= r_i - r_c$, where r_c is the radius of the column or post at the centre.

$\overline{G}$ (inner going) $= 2(r_c - 270°)\sin\theta/2$

Clear headroom: $2H_i = h_1(\phi/\theta) - t_L$, where ϕ = the angle of rotation at a distance along radius, θ = the angle of taper of tread, h_1 = the rise, t_L = the thickness of landing.

Minimum splayed straight length $L = B' + 2/3 \times$ bearing.

are involved, the data for helical stairs circular in plan are modified to include the geometry of the inclined straight arms. The data collected are from countries such as Britain, Spain, USA, Germany, Sweden, Pakistan, India, Italy, Turkey and Japan.

1.3 LOADS AND LOAD COMBINATIONS

Loads and their combinations vary from one country to another. The partial safety factors associated with these loads vary as well and they largely depend on whether the stairs are analysed by the elastic, limit state, strength reduction and other concepts. In general, it is easy to compute dead loads and loads due to self weight and finishes. The disagreements are on the imposed loads (3 kN/m^2 to 5 kN/m^2) and the partial safety factors for loads and materials. Several examples in the text will indicate this dilemma. The general opinion is that steps should be loaded also with concentrated loads. The British practice is to check individual treads by placing on them two loads of 0.9 kN at 300 mm spacing and placed symmetrically about the centre line of the tread. For details individual codes shall be consulted.

1.4 MATERIALS AND STRESSES

For materials and their allowable stresses, individual codes are referred to. In the absence of such codes(s), Table 1.4 should be consulted for the preliminary analysis and design of staircases.

1.5 ADDITIONAL SPECIFICATIONS FOR THE REINFORCEMENT OF CONCRETE STAIRS

1.5.1 *Reinforcement size*

A standard range of bars and sizes is available for use in reinforced concrete. They may be hot-rolled (mild steel, high yield steel) or cold worked (high yield steel). Bars are made in a range of diameters from 8 to 40 mm. Special sizes of 6 and 50 mm are seldom available. The specification for steel covers chemical composition. Tensile strength, ductility, bond strength, weldability and cross-section area can be found in various codes.

1.5.2 *Fabric*

Fabric reinforcement is manufactured to BS4483 and to ASTM 1992 requirements. There are four types of fabric made from hard drawn mild steel wire of $f_y = 485$ N/mm^2 or from cold-worked high yield bars:

a) *Square mesh fabric*: regular bars of lightweight (A type). They are used in walls and slabs.

b) *Structural fabric*: main wires 100 mm crs (B type) cross section of wires 200 mm crs.

c) *Long mesh fabric*: main wires 100 mm crs (C type) cross wires 400 mm crs.

d) *Wrapping fabric*: lightweight square mesh (D type) encased conditions or fire resistance main wire cross-sectional area 252 mm^2, $f_y =$ 250 N/mm^2.

Table 1.5 gives the necessary data for bars and fabric reinforcement.

Table 1.4. Materials and stresses: steel, concrete, aluminium and timber.

Country	Steel	Concrete	CI
UK	a) steel bars hot rolled: $f_y = 250$ N/mm^2 high yield: $f_y = 460$ N/mm^2 $E_s = 200$ GN/m^2; $v = 0.3$ b) steel sections and plates – varies	C25 to C40 grade $f_c = 25$ N/m^2 to 40 N/m^2 E_c(average) = 20 N/m^2 $v = 0.15$ to 0.2 $f_t = 1/10$ of C in general unit mass = 2400 kg/m^3 = 23.6 kN/m^3	BS5328 BS5950 BS8110 BS648
USA	a) steel bars (psi) grade yield stress $f_y = 40{,}000$; 70,000; 60,000; 90,000 b) steel sections and plates Typical steel (values in Ksi) A36-30-36; 48-60 A572-42-65; 63-70 A572-42-65; 75-80 $v = 0.3$; G = shear modulus = 11,000 Ksi $E_s = 30 \times 10^6$ (psi) (200 kN/mm^2)	f_c – cylindrical concrete strength 3000, 4000 (psi); $E_c = 3 \times 10^6$ (psi) (20 kN/mm^2) $v = 0.15$ to 0.2	ACI318 AISC Specifications 1995 ASTM Specifications 1995
EUROPE	a) steel bars S220; S400; S500 b) steel sections and plates Fe360; Fe430 to Fe510 yield stress f_v 235 N/mm^2 to 332 N/mm^2 Ultimate stress $f_u = 360$ N/mm^2 to 470 N/mm^2 $E_s = 200$ GN/m^2; $v = 0.3$	C12/15 to C50/60 $f_{cu} = 12$ N/mm^2 to 50 N/mm^2 $f_t = 1.6$ N/mm^2 to 4.1 N/mm^2 $E_c = 20$ GN/m^2 $v = 0.1$-0.2	Eurocode 2 DDENV 1992 Eurocode 3 DDENV 1993 EN 10025
CANADA	steel bars N/mm^2 grade 300 350 400 $f_s = 400$ 350 400 $f_s = 400$ 550 600 $E_s = 200$ GN/m^2 $v = 0.3$ steel sections and plates same as USA	$f_c = 20$ N/mm^2 or 30 N/mm^2 $E_c = 20$ GN/mm^2 $v = 0.1$ to 0.2	CSA A23.3 M84

Table 1.4 (cont.).

Country	Aluminium	Timber			CI
UK	same as USA	class S_1 to S_7 – 37.5 to 15 N/mm^2 shear 2 to 4 N/mm^2; class SC3 and SC4 σ bending = 7.5 N/mm^2 c compression = 2.4 N/mm^2 G_{shear} = 0.7 N/mm^2 E_{mean} = 9900 N/mm^2 E_{min} = 6600 N/mm^2 hardwood 75 × 75 post σ bending = 18.1 N/mm^2 E = 13,600 N/mm^2 G 660 to 570 N/mm^2			BS5268
USA	6061-H116 c = 140 N/mm^2 t = 150 N/mm^2 G shear = 105 N/mm^2 f_y (yield) = 286 N/mm^2 σ bending = 133 N/mm^2	t = 483 N/mm^2; f_y = 370 N/mm^2 t = 323 N/mm^2; f_y = 308 N/mm^2 t = 590 N/mm^2; f_y = 542 N/mm^2			ASTM 2024-7351* 6061-7651† 7075-7651‡
EUROPE	Not available		hardwood = C14 to C40	softwood = D30 to D70	Eurocode 5 EC5
		$f_{m,k}$	= 14 to 40	= 30 to 70	
		$f_{t,o,k}$	= 8 to 24	= 18 to 42	
		$f_{c,o,k}$	= 16 to 26	= 23 to 24	
		$f_{v,k}$	= 1.7 to 3.8	= 3.0 to 6.0	
		$E_{o\,mean}$	= 7000 to 14,000	= 10,000 to 20,000	
		G_{mean}	= 440 to 880	= 600 to 1250	
CANADA	same as USA	σ bending = 11.6 to 8.58 N/mm^2 c compression = 7.99 to 5.04 N/mm^2 H shear = 0.36 to 0.38 N/mm^2 E = 10,550 to 4150 N/mm^2 G = 660 to 570 N/mm^2 v = 0.1 to 0.3			CSA 0121-M

f_c = compressive stresses; f_t = t tensile stress; CI = Code identification; C (Cube strength) = f_{cu}; * σ = stress in bending; † t = tensile stress; ‡ f_y = stress at yield. For all the following, values are in N/mm^2: $f_{m,k}$ = stress in bending; $f_{t,o,k}$ = tension // to grain; $f_{c,o,k}$ = compression // to grain; $f_{v,k}$ = shear; $E_{o\,mean}$ = mean Young's Modulus // to grain; G_{mean} = mean shear modulus.

1.5.3 *Cover to reinforcement*

The distance between the outermost bars and the concrete face is termed the cover. The cover provides protection against corrosion, fire and other accidental loads. For the bond to be effective cover is needed. Various concrete codes allow grouping or bundling of bars. In this case the perimeter around a bundle determines the equivalent area of a 'single bar'. The cover also depends on the grade of concrete and the full range of exposure conditions.

Table 1.5. Bars and fabric reinforcement with concrete cover.

	Bar designation													
Britain, Europe, Japan and Russia bar types (mm)	6	8	10	12	16	20	–	25	–	32	–	40	–	–
USA, Canada, S. America bar types (mm) denoted by #	–	–	#3	#4	#5	#6	#7	#8	#9	#10	#11	–	#14	#18
Area (mm^2)	28	50	78	113	201	314	387	491	645	804	1006	1257	1452	2581

Mesh type	Size of wires (mm)		Area (mm^2)		Weight (kg/m^2)
	main	cross	main	cross	
1. Square mesh fabric (200×200)					
A393	10	10		393	6.16
A252	8	8		252	3.95
A193	7	7		193	3.02
A142	6	6		147	2.22
A98	5	5		98	1.54
2. Structural fabric (100×200)					
B1131	12	8	1131	252	10.90
B785	10	8	785	252	8.14
B503	8	8	503	252	5.93
B386	7	7	385	193	4.53
B283	6	7	283	193	3.73
B196	5	7	196	193	3.05
3. Long mesh fabric (100×400)					
C785	10	6	785	70.8	6.72
C636	9	6	636	70.8	5.55
C503	8	5	503	49.0	4.34
C385	7	5	385	49.0	3.41
C283	6	5	283	49.0	2.61
4. Wrapping fabric	2.5	2.5			
D49 (100×100)			49	49	0.76
D98 (200×200)			98	98	1.54

Conditions of exposure	Nominal cover (mm)				
Mild	25	20	20**	20**	20**
Moderate	–	35	30	25	20
Severe	–	–	40	30	25
Very severe	–	–	50$^+$	40***	30
Extreme	–	–	–	60***	50
Water/cement ratio	0.65	0.60	0.55	0.50	0.45
Concrete grade	C30	C35	C40	C45	C50

*All values in the table are for h_{agg} maxsimum aggregate size of 20 mm.
**To be reduced to 15 mm provided $h_{agg} > 15$ mm.
***Air entrainment should be used when concrete is subject to freezing.
$^+$ Could be increased further if needed.

CHAPTER 2

Structural analysis of staircases: Classical methods

2.1 INTRODUCTION

This chapter deals with classical methods of analysis. The author has reproduced these methods clearly by adopting uniform symbols. Wherever possible the reader is given a positive basis for understanding these methods by explaining the basic philosophy of each method and the assumptions associated with it. Examples are given of these methods so that students and practising engineers can easily translate them into practical problems. Most of these methods are based on the Strain Energy concept.

2.2 METHODS FOR FREE STANDING STAIRS

These are the following: Bangash Generalised Method based on Gould's (1963) numerical solution (uniform loads with various boundary conditions); Taleb's Method (1964) symmetrical and asymmetrical loads; Methods of Space Intersections of Plates; Liebenberg Method (1956, 1960, 1962) and Siev's Method (1962, 1963).

2.3 METHODS FOR HELICAL STAIRS

For helical stairs, two methods are used: Morgan's Method (1960) and Cohen's Method (1955).

2.4 A GENERALISED ANALYSIS OF A CANTILEVER STAIRCASE

The author has developed a generalised analysis based on the original work done by Gould P. (*Journal of the American Concrete Institute*, 1963) which is summarised in Section 2.4.2. Here the Strain Energy principle is adopted. The staircase is considered as a frame and the moment at the intermediate landings is transferred between the legs by torsion developed through the landing. This method depends on the type of support conditions at the upper landing. In order that the staircase behaves as a frame, vertical and horizontal forces must also be transmitted between the legs of the staircase through the landing. These should act through the centre line of the landing parallel to the longitudinal axis of the legs and are eccentric on the legs. The additional bending and torsional moments at the intersection of flights and landing have only a minor effect on the design and are thus ignored.

2.4.1 *Notation for the analysis*

b = width of the supporting beam;
H = horizontal reaction;
H_1, H_2 = heights;
E = modulus of elasticity (Young's modulus);
K_H = horizontal spring constant
K_M = rotational spring constant
K_V = vertical spring constant
(kN/m or lb, kip/ft;)
L = length;
M = bending moment;
R = reaction;
T = torsional moment;
U = strain energy.

2.4.2 *Gould's method (July 1963)*

Notation for the analysis

b = width of intermediate landing, in;
b' = long dimension of the tie or hoop, in;
c = width of supporting beam, in;
d = depth of stair slab, in;
e = M_P/P measured from centroid of base, in;
e' = distance from centroid of footing to line of action of P', in;
f_s = allowable stress in reinforcement, psi;
h = depth of intermediate landing, in;
h' = short dimension of the tie or hoop, in;
s = spacing of ties in landing, in;
t = depth of supporting beam, in;
w = width of stair slab, ft;
A_{sl} = area of horizontal steel perpendicular to ties in intermediate landing, sq in;
A_{sv} = area of all shear reinforcement at a given section in the intermediate landing, i.e. the area of two bars in the hoop, sq in;
C_1 = $642 + EI/1.3K_V$;
C_2 = $875 + EI/1.3K_H$;
C_3 = $15.83 + EI/1.3K_M$;
E = modulus of elasticity, kips per sq in;
F = ratio of actual length to horizontal projection;

F_1 = integration factor;
F_2 = integration factor;
G = shearing modulus, kips per sq in;
H_n = horizontal reaction at nth support, kips;
I = moment of inertia of the stair slab, in^4;
I_{bx} = moment of inertia of the supporting beam, about the horizontal axis, in^4;
I_{by} = moment of inertia of the supporting beam, about the vertical axis, in^4;
K_H = horizontal spring constant of support, kips per ft;
K_M = rotational spring constant of support, ft-kips per radian;
K_V = vertical spring constant of support, kips per ft;
L = length of the supporting beam, ft;
M_D = bending moment at base of staircase, ft-kips;
M_{mn} = bending moment at Point m in Member mn, ft-kips;
P = axial load on the base of the staircase, kips;
P' = equivalent axial load applied to the footing at an eccentricity e', kips;
T = torsional moment at intermediate landing, ft-kips;
U = strain energy due to bending, ft-kips;
V_n = vertical reaction at nth support, kips;
V_{mn} = shear at m in Member mn, kips;

Greek

$\left.\begin{matrix}\alpha\\ \beta\\ \lambda\end{matrix}\right\}$ elastic torsion theory constants for rectangular sections;
τ = angle of twist per unit length, radians;
ϕ = total angle of twist, radians;
Δ_{H_n} = horizontal deflection of nth support, in;
Δ_{V_n} = vertical deflection of nth support, in;
Δ_{M_n} = rotation of nth support, radians;
ω = torsional shear stress, psi.

Torsion at intermediate landing

The maximum torsional shear stress can be approximated by the formula:

$$\omega = \frac{T}{\alpha b h^2}$$

The coefficient α is itself proportional to b/h but approaches a limit of 0.333 for large values of b/h.

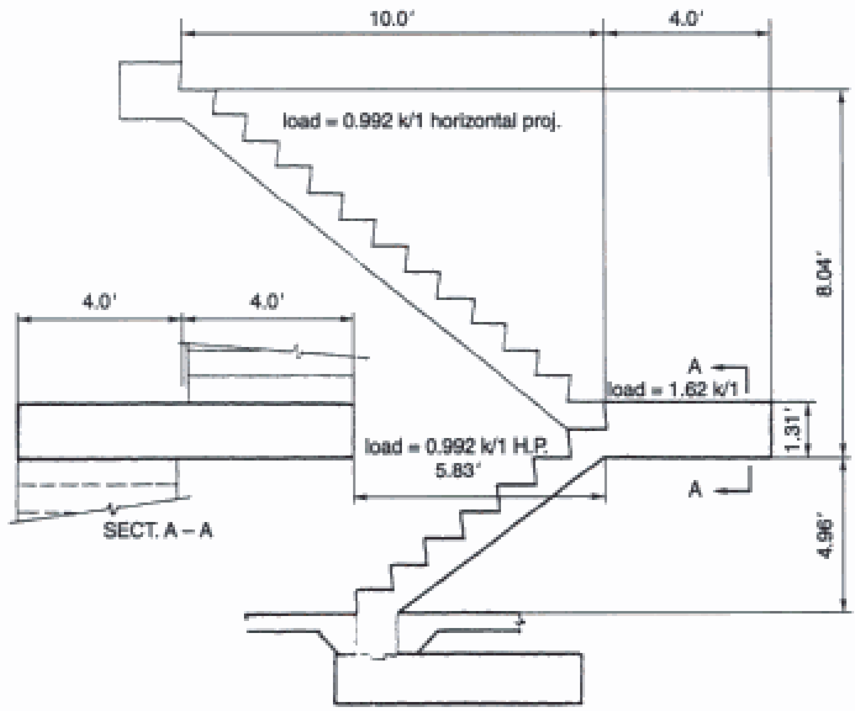
$$\sum M_B$$

$$M_{BA} = T - \frac{\omega b g^2}{2}$$

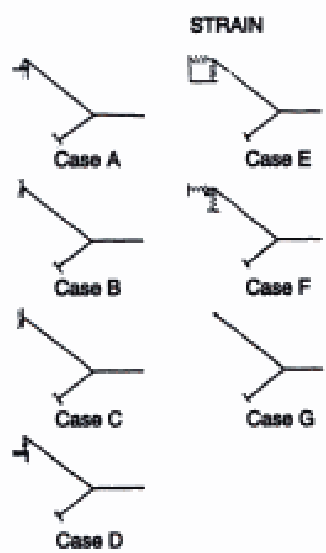
$$M_{BC} = T - \frac{\omega b g^2}{2}$$

$$M_{BA} + M_{BC} = 2T$$

$$T = \frac{M_{BA} + M_{BC}}{2}$$



Elevation of cantilever staircase showing dimensions and loads



Support conditions for Cases A through D

Case A – vertical reaction at point A

$$\frac{\partial U}{\partial V_A} = \frac{1}{EI}\int_0^{10} (V_A x - 0.496x^2)x\,\mathrm{d}x$$

$$+\frac{1}{EI}\int_0^{5.83} \left[0.496x^2 + 16.4x - 36.6 + V_A(10-x)\right][10-x]\,\mathrm{d}x$$

$$= 0$$

After the necessary integration has been performed:

$$\frac{\partial U}{\partial V_A}EI = -856 + 642V_A = 0$$

Hence,

$V_A = 1.33$ kips

$M_D = 76.0 + 4.17 \times 1.33 = 81$ ft-kips

$V_D = 22.18 - 1.33 = 20.85$ kips

$M_{BA} = 10 \times 1.33 - 49.6 = 36.3$ ft-kips (clockwise)

$M_{BD} = 10 \times 1.33 - 36.6 = 23.3$ ft-kips (anticlockwise)

$T = \dfrac{23.3 + 36.3}{2} = 29.8$ ft-kips

Case B – horizontal and vertical reaction at point A

$$\sum M_B = -49.6 + 11.7 - 8.04H_A + 10V_A = 0$$

$$\sum M_D = -76.1 - 3.6 + 4.17V_A - 13H_A = 0$$

$H_A = 9.10$ kips

$V_A = 11.08$ kips

$V_D = 22.18 - 11.08 = 11.10$ kips

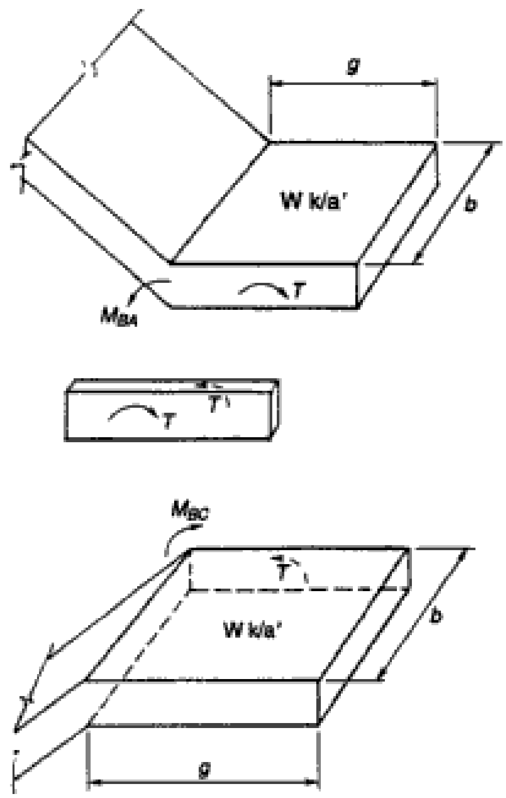
$M_{BA} = 11.7$ ft-kips (clockwise)

$M_{BD} = 1.3$ ft-kips (clockwise)

$T = \dfrac{11.7 - 1.3}{2} = 5.2$ ft-kips

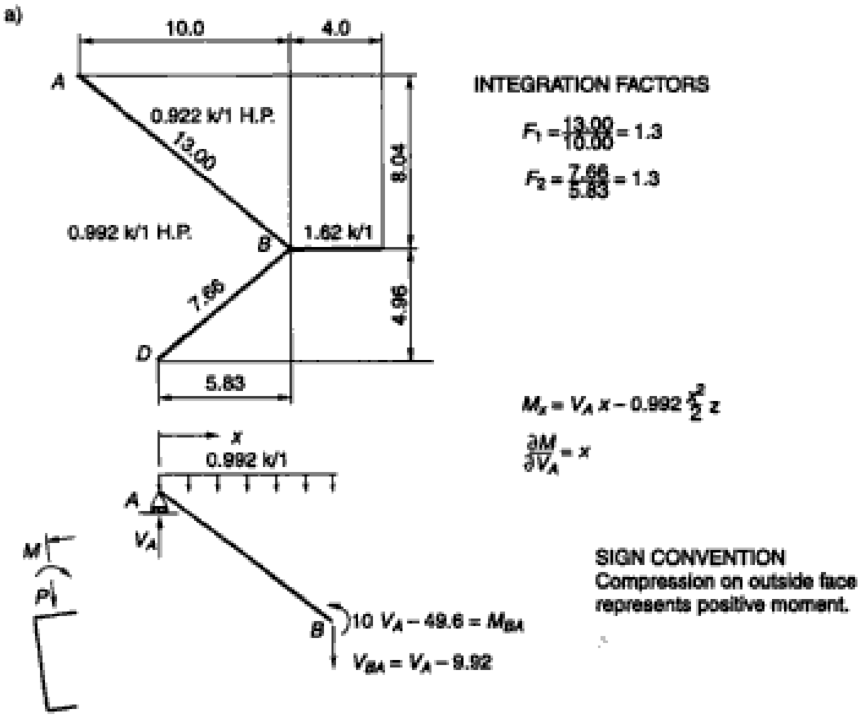
Summary of reactions and moments.

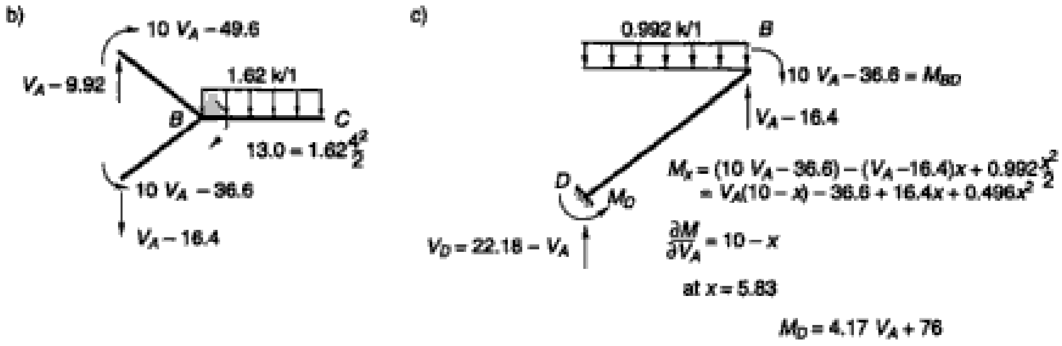
Case	V_A, kips	V_P, kips	H, kips	M_A, ft-kips	M_P, ft-kips (anticlockwise)	M_{BA}, ft-kips (clockwise)	M_{BD}, ft-kips (anticlockwise)	T, ft-kips
A	1.33	20.85	–	–	81.00	36.30	23.30	29.80
B	11.08	11.10	9.10	–	3.60	11.70	1.30	5.20
C	12.00	10.08	8.89	7.90	2.20	8.98	4.02	2.48
D	1.27	20.91	–	–	81.30	36.90	23.90	30.40
E	6.35	15.83	4.95	–	38.00	25.90	12.90	19.40
F	6.59	15.59	5.02	1.40	36.90	25.40	12.40	18.90
G	–	22.18	–	–	76.00	49.60	36.60	43.10



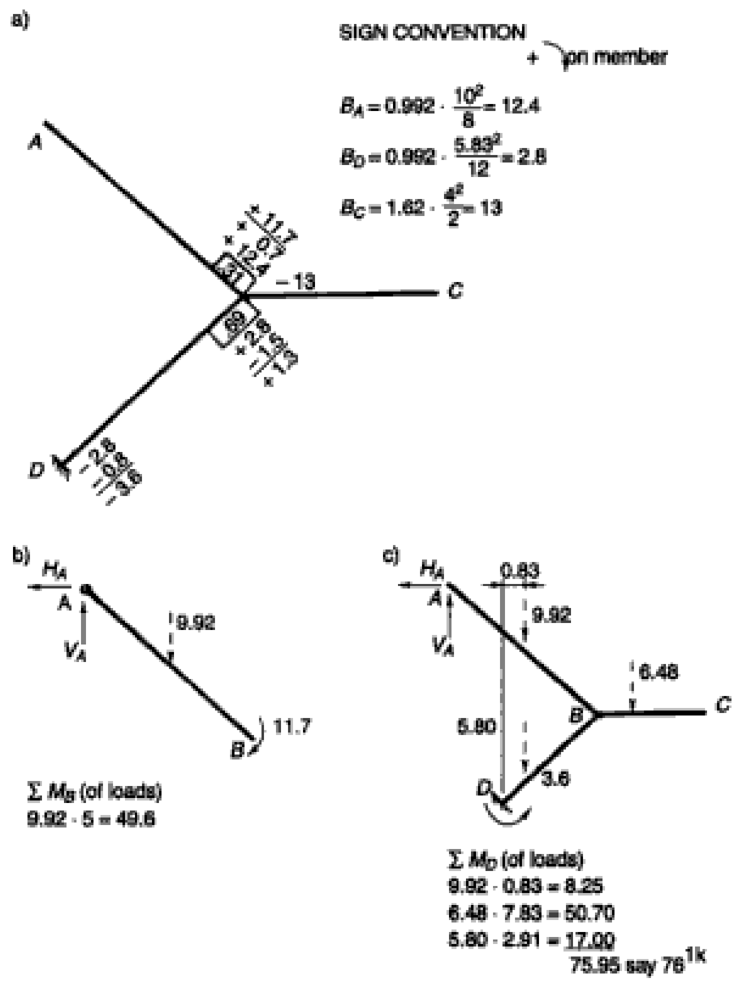
Moment acting on landing

a)





Free-body diagrams of staircase members assuming a vertical reaction at Support A



Case C – fixed support at point A

The method of analysis is exactly the same as that of Case B so only the results are presented.

$$V_A = 12.0 \text{ kips}$$
$$V_D = 10.08 \text{ kips}$$
$$H_A = 8.89 \text{ kips}$$
$$M_{BA} = 8.98 \text{ ft-kips}$$
$$M_{BD} = 4.02 \text{ ft-kips}$$
$$T = \frac{8.98 - 4.02}{2} = 2.48 \text{ ft-kips}$$

Case D – vertical reaction of flexible support at point A

This case is similar to Case A except that the support at point A is flexible. Assume that the support may be represented by a spring such that:

$$K_V = \frac{V_A}{\Delta V_A}$$
$$\frac{\partial U}{\partial V_A} = -\frac{V_A}{K}$$
$$(-856 + 642V_A)1.3 = -\frac{V_A}{K_V}EI$$
$$V_A = \frac{856}{642 + \dfrac{EI}{1.3K_V}}$$
$$V_A = \frac{856}{C_1}\cdots; \quad C_1 = 642 + \frac{EI}{1.3K_V}$$

Case E – vertical and horizontal reactions on flexible supports at point A

This case is similar to Case B. However, since deflections are involved, the method of solution will be different. The supports will be treated as springs with constants K_H and K_V.

Castigliano's theorem will then become:

$$\frac{\partial U}{\partial V_A} = -\frac{V_A}{K_V}$$
$$\frac{\partial U}{\partial H_A} = -\frac{H_A}{K_H}$$

$$\frac{\partial U}{\partial V_A} = \frac{1}{EI}\int_0^{10} (V_A x - 0.804 H_A x - 0.496x^2)x\,dx$$

$$+\frac{1}{EI}\int_0^{5.83} \big[0.496x^2 + 16.4x - 36.6$$

$$+ V_A(10-x) + H_A(-8.04 - 0.85x)\big](10-x)\,dx$$

$$= -\frac{V_A}{K_H}$$

$$\frac{\partial U}{\partial V_A} = \frac{1.3}{EI}(-870 - 689H_A + 642V_A) = -\frac{V_A}{K_V}$$

and

$$\frac{\partial U}{\partial H_A} = \frac{1.3}{EI}(-870 - 689H_A + 642V_A) = -\frac{H_A}{K_H}$$

where,

$$H_A = \frac{598{,}000 + 310C_1}{C_1C_2 - 474{,}000}$$

$$V_A = \frac{213{,}000 + 870C_2}{C_1C_2 - 474{,}000}$$

$$C_2 = 875 + \frac{EI}{1.3K_H}$$

$$\frac{\partial U}{\partial H_A} = \frac{1}{EI}\int_0^{10} (V_A x - 0.804H_A x - 0.496x^2)(-0.804x)\,dx$$

$$+\frac{1}{EI}\int_0^{5.83} \big[0.496x^2 + 16.4x - 36.6 + V_A(10-x)$$

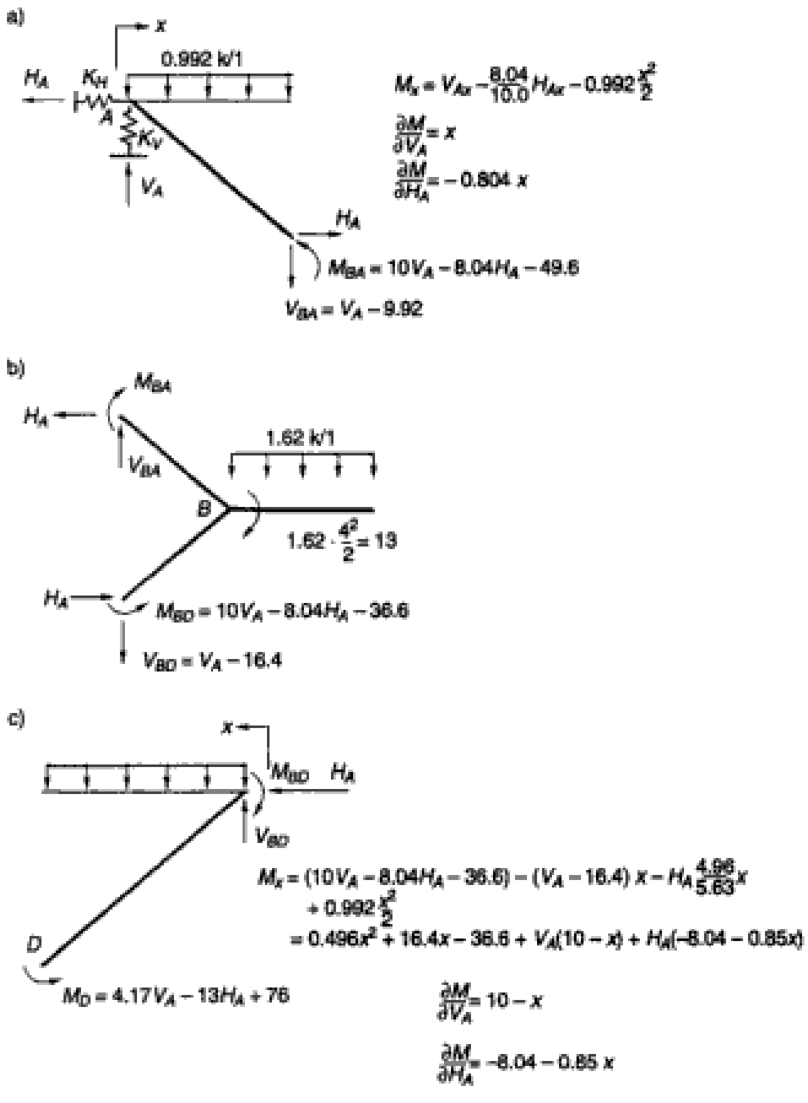
$$+ H_A(-8.04 - 0.85x)\big][-8.04 - 0.85x]\,dx$$

$$= -\frac{H_A}{K_H}$$

Case F – partial fixity at point A

If restraint to rotation proportional to the angle of twist is assumed at point A ($K_M = M/\Phi$), the effect of the moment may be accounted for in a similar manner as the elastic deflections of the supports. The equations of Case E – may easily be modified by the addition of a M_A-term to the moment expressions. An additional equation is obtained from this condition.

$$\frac{\partial U}{\partial M_A} = -\frac{M_A}{K_M}$$



Free-body diagrams of staircase members assuming a flexible vertical and horizontal reaction at Support *A*

The three equations are then

$$\frac{\partial U}{\partial M_A} = -\frac{1}{EI}\int_0^{10} \left(V_A x - 0.804 H_A x - 0.496x^2 - M_A\right)(-1)\,\mathrm{d}x$$

$$+ \frac{1}{EI}\int_0^{5.83} \big[0.496x^2 + 16.4x - 36.6 + V_A(10 - x)$$

$$+ H_A(-8.04 - 0.85x) - M_A\big](-1)\,\mathrm{d}x$$

$$= -\frac{M_A}{K_M}$$

$$\frac{\partial U}{\partial V_A} = \frac{1}{EI}\int_0^{10} \left(V_A x - 0.804 H_A x - 0.496x^2 - M_A\right)x\,\mathrm{d}x$$

$$+\frac{1}{EI}\int_0^{5.83} \big[0.496x^2 + 16.4x - 36.6 + V_A(10-x)$$

$$+ H_A(-8.04 - 0.85x) - M_A\big](10-x)\,\mathrm{d}x$$

$$= -\frac{V_A}{K_V}$$

$$\frac{\partial U}{\partial H_A} = \frac{1}{EI}\int_0^{10} \left(V_A x - 0.804 H_A x - 0.496x^2 - M_A\right)(-0.804x)\,\mathrm{d}x$$

$$+\frac{1}{EI}\int_0^{5.83} \big[0.496x^2 + 16.4x - 36.6 + V_A(10-x)$$

$$+ H_A(-8.04 - 0.85x) - M_A\big](-8.04 - 0.85x)\,\mathrm{d}x$$

$$= -\frac{H_A}{K_H}$$

After the necessary integrations have been performed, the equations simplify into the following expressions.

$$\frac{1.3}{EI}(66.6 - 91.3V_A + 101.4H_A + 15.83M_A) = -\frac{M_A}{K_M}$$

$$\frac{1.3}{EI}(-870 + 642V_A - 689H_A - 91.3M_A) = -\frac{V_A}{K_V}$$

$$\frac{1.3}{EI}(-310 + 688V_A + 875H_A + 101.5M_A) = -\frac{H_A}{K_H}$$

From these expressions we obtain:

$$V_A = \frac{-7{,}210{,}000 + 870C_2C_3 + 214{,}000C_3 - 6080C_2}{12{,}750{,}000 - 10{,}300C_1 - 8330C_2 - 474{,}000C_3 + C_1C_2C_3}$$

$$H_A = \frac{14{,}850{,}000 + 6780C_1 + 598{,}000C_3 + 310C_1C_3}{12{,}750{,}000 - 10{,}300C_1 - 8330C_2 - 474{,}000C_3 + C_1C_2C_3}$$

$$M_A = \frac{-9{,}900{,}000 - 31{,}500C_1 + 79{,}500C_2 - 66.6C_1C_2}{12{,}750{,}000 - 10{,}300C_1 - 8350C_2 - 474{,}000C_3 + C_1C_2C_3}$$

where,

$$C_3 = 15.83 + \frac{EI}{1.3K_M}$$

The values for reactions and moments are summarized in the table in Case E. The results obtained, considering the partial fixity against rotation, are similar to that of Case E, indicating that the added restraint has only a small influence on the moments and reactions.

Case G – free end at point A

The results may be obtained from statistics and are tabulated in the table in Case E. This case is presented to show the influence of the various restraints on the upper leg.

Properties of supporting members

Supporting beam. For Cases D, E and F a supporting beam is assumed with the following dimensions and section properties.

Size: $c = 12$ in; $t = 18$ in; $L = 20$ ft simple span restrained against twist at ends; $E = 3 \times 10^3$ kips per sq in; $G = 1 \times 10^3$ kips per sq in; $I_{bx} = 5830$ in^4; $I_{by} = 2590$ in^4

$$K_H = \frac{48EI_{by}}{L^3} = 324 \text{ kips per ft}$$

$$K_V = \frac{48EI_{bx}}{L^3} = 732 \text{ kips per ft}$$

$$\textit{Torsion} = \tau\beta t c^3 G$$

The 20-ft beam will act as two 10-ft cantilevers fixed against rotation:

$$\tau = \frac{\Phi}{L} = \frac{1}{20}$$

$t/c = 1.5$; $\beta = 0.196$ and $K_M = 4240$ ft-kips per radian for each 10-ft cantilever or 8480 ft-kips per radian total.

Staircase:

$$w = 4 \text{ ft}, \quad d = 6.5 \text{ in}$$

$$I = \frac{48 \times 6 \times 5^3}{12} = 1100 \text{ ft}^4$$

Constants:

$$C_1 = 642 + \frac{22{,}900}{1.3 \times 732} = 666 \text{ ft}^3$$

$$C_2 = 875 + \frac{22{,}900}{1.3 \times 324} = 929 \text{ ft}^3$$

$$C_3 = 15.83 + \frac{22{,}900}{1.3 \times 4240} = 20.00 \text{ ft}^3$$

Typical design (Case A)

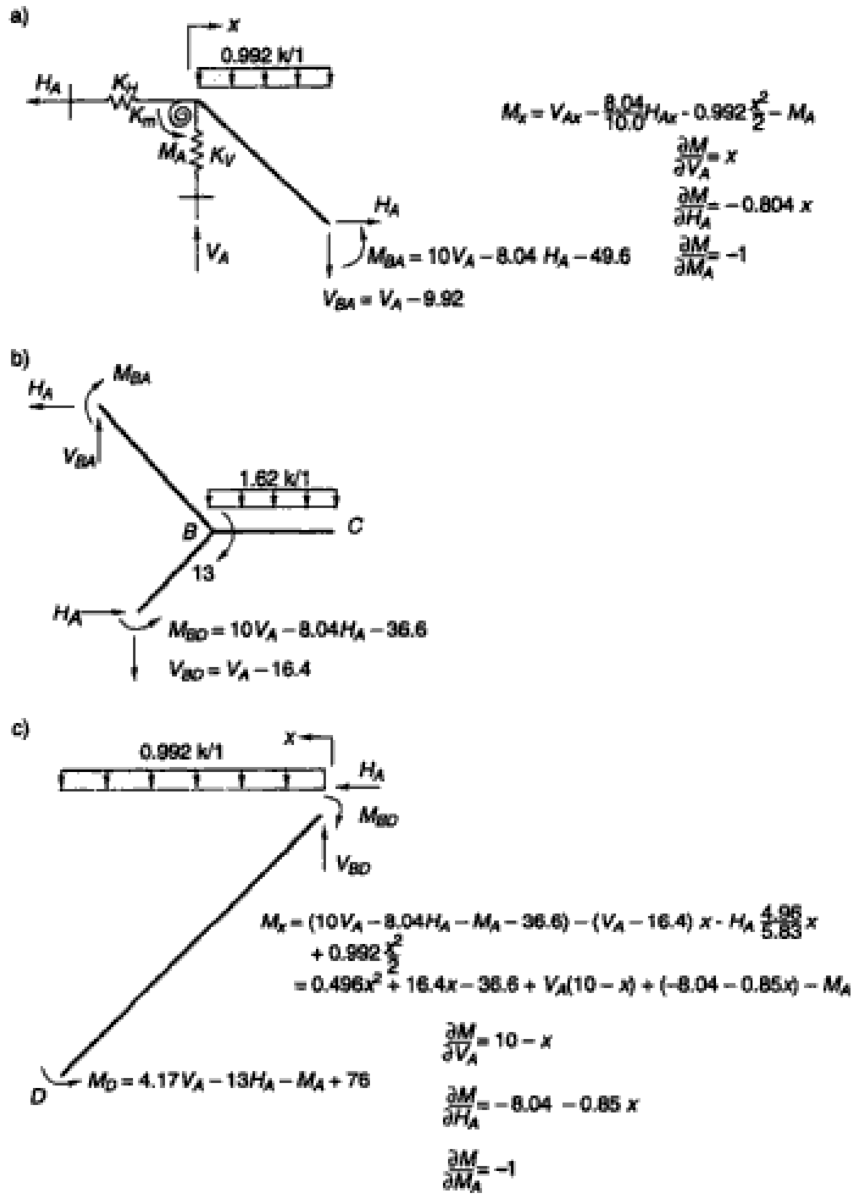
Torsional reinforcement

From Figure 5b and Reference 3

$$T = 29.8 \text{ ft-kips}$$

$$b/h = 3.04$$

$$\alpha = 0.267\lambda = 0.845$$



$$\omega = \frac{1}{0.267}\left(\frac{29{,}800 \times 12}{15.75^2 \times 48}\right) = 113 \text{ psi}$$

Ties

$$A_{sv} = \frac{Ts}{\lambda f_s b' h'}$$

Then,

$$s = \frac{\lambda A_{sv} f_s b' h'}{T}$$

For #4 ties

$A_{sv} = 2 \times 0.20 = 0.40$ sq in

$$s = \frac{0.845 \times 0.40 \times 20{,}000 \times 45 \times 12}{29{,}800 \times 12} = 10.4 \text{ in}$$

Hence, for design use #4 ties at 8 in.

Horizontal bars

For horizontal bars, an equal volume of steel is provided.

$$A_{sl} = A_{sv}\left(\frac{b' + h'}{s}\right) = \frac{0.40}{8}(45 + 12) = 2.86 \text{ sq in}$$

Hence, for design use 10 #5 = 3.10 sq in.

Additional reinforcement will be required near the junction for one-half the torsional moment:

$b/h = 1.0 \qquad b' = 12$ in

$\lambda = 0.835 \qquad h' = 12$ in

Ties

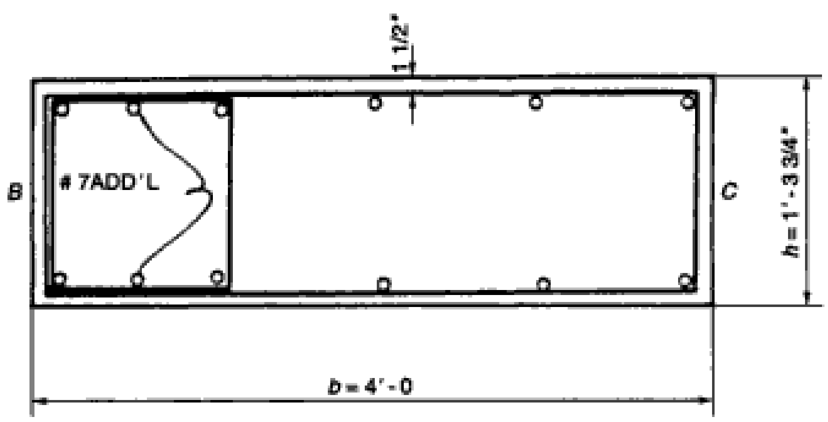
$$A_{sv} = \frac{Ts}{\lambda f_s b' h'} = \frac{0.5 \times 29{,}800 \times 12 \times 8}{0.845 \times 20{,}000 \times 12 \times 12} = 0.60 \text{ sq in}$$

This reinforcement is provided by #4 bars at 8 in alternate spacing.

Horizontal bars

$$A_{sl} = \frac{0.40}{8} \times 24 = 1.20 \text{ sq in}$$

For design use 2 #7 bars.



Typical torsional reinforcement

b = 48"
h = 15.75"
b' = 45"
h' = 12"

2.4.3 *Case studies: Bangash generalised analysis based on Gould (July 1963)*

Case I: vertical reactions at D

$$\frac{\partial U}{\partial R_D} = \partial_D = \int \frac{M\,\mathrm{d}x}{EI}\frac{\partial M}{\partial R_D} \tag{2.1}$$

since

$$U = \frac{1}{2}\int \frac{M^2\,\mathrm{d}x}{EI} \tag{2.2}$$

$$M_x = \text{moment at a distance } x = R_D x - \omega_1 \frac{x^2}{2} \tag{2.3}$$

Figure 2.1(a) $R_B = R_D - \omega_1 L_2 \quad R_D = R_B + \omega_1 L_2$ (2.4)

$$M_{BA} = R_D L_2 - \frac{\omega_1 L_2^2}{2} \tag{2.5}$$

Figures 2.1(b) and (b_1)

reaction at D:

$$\left.\begin{aligned} & R_D - \omega_1 L_2 \\ & M_{DB} = R_D L_2 - \frac{\omega_1 L_2^2}{2} \\ & M_B = \frac{\omega_2 L_3}{2} \\ & M_{AB} = R_D L_2 - \frac{1}{2}(\omega_1 L_1^2 - \omega_2 L_3^2) \end{aligned}\right\} \tag{2.6}$$

reaction at A:

$$M_{AB} = R_D L_2 - \frac{1}{2}(\omega_1 L_1^2 - \omega_2 L_3^2) = V_D - (\omega_3 L_2 + \omega_2 L_3) \tag{2.7}$$

$$\begin{aligned} M_x = {} & \left[R_D L_2 - \left\{\frac{\omega_1 L_2^1}{2} - \frac{\omega_2 L_3^2}{2}\right\}\right] - \left[R_D - \{\omega_1 L_2 + \omega_2 L_3\}\right] \\ & + \frac{\omega_1 x^2}{2} \end{aligned}$$

$$R_D(L_2 - x) - \left[\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2}\right] = \frac{1}{2}\frac{\omega_1 x^2}{2} \tag{2.8}$$

$$\frac{\partial M_x}{\partial R_D} = L_2 - x \tag{2.9}$$

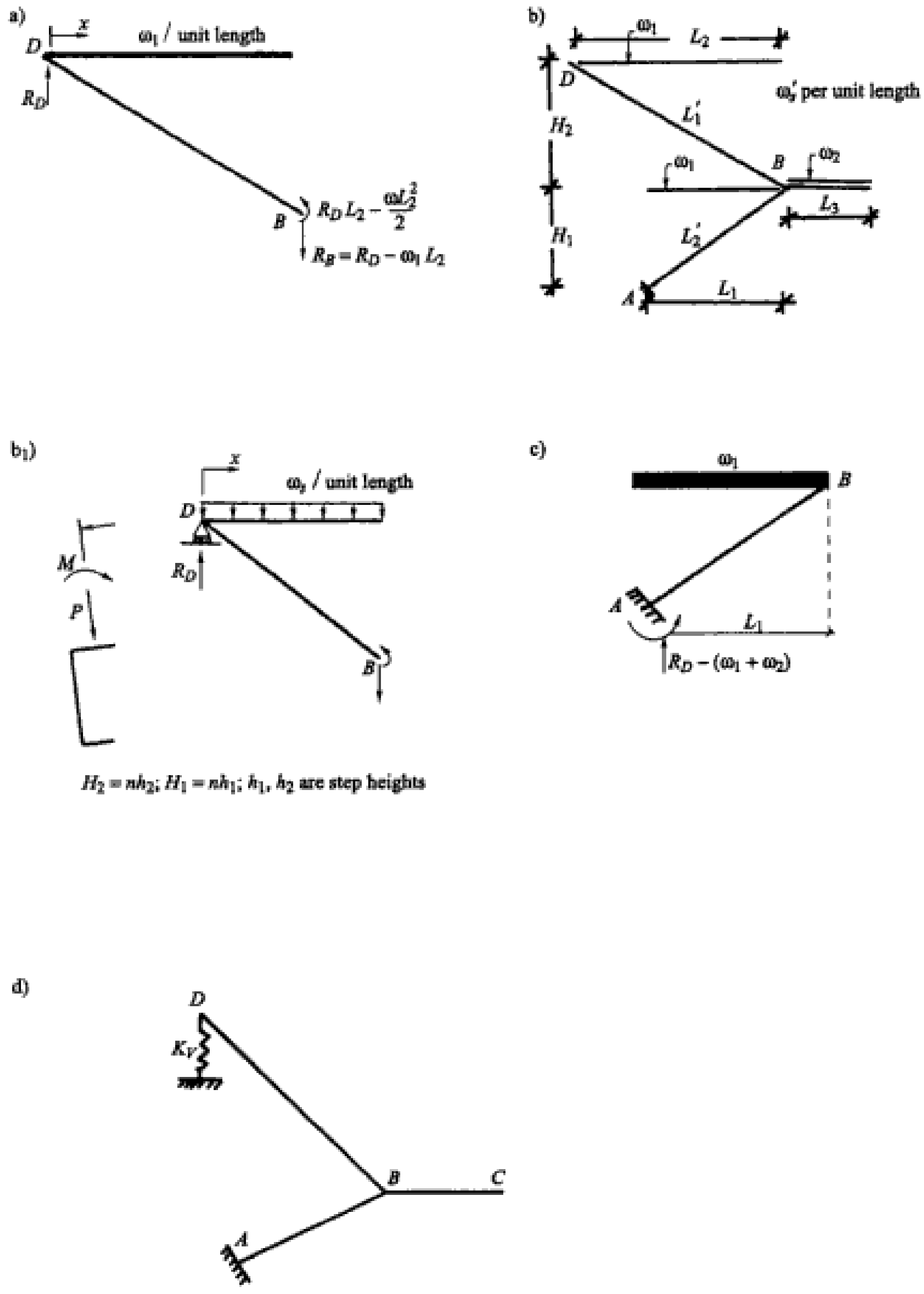


Figure 2.1. A cantilever staircase under loads.

M_{AB} from the above equation is written as $(x = L_1)$

$$M_{AB} = (L_2 - L_1)R_D - \left[\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2}\right] + \frac{1}{2}\frac{\omega_1 x^2}{2}$$

$$= R_D(L_2 - L_1) - \frac{1}{2}\left[\omega_1 L_1^2 - \omega_2 L_3^2 + \frac{\omega_1 x^2}{2}\right] \tag{2.10}$$

since,

$$\frac{\partial U}{\partial R_D} = \frac{1}{EI}\int_0^{L_2}\left(R_D x - \frac{1}{2}\frac{\omega_1 x^2}{2}\right)x\,\mathrm{d}x$$
$$+ \frac{1}{EI}\int_0^{L_1}\left[\frac{\omega_1 x^2}{4} + (\omega_1 + \omega_2)x - \left\{\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2} - \right\}\right]$$
$$+ R_D(L_2 - x)^2\,\mathrm{d}x = 0$$

$$EI\frac{\partial U}{\partial R_D} = L_2^3\left\{-\frac{\omega_1 L_2}{16} + \frac{R_D}{3}\right\} + L_1^3\left\{\frac{\omega_1}{12} - \frac{\omega_1}{2} + \frac{1}{3}\right\}$$
$$+ \frac{L_1^2}{2}\{\omega_1 + \omega_2 - 2L_2\} + L_1\left\{\frac{\omega_2 L_3^2}{2} + L_2^2\right\} = 0 \tag{2.11}$$

but

$$EI\frac{\partial U}{\partial R_D} = \frac{R_D}{3}(L_3)^3 - \left(\frac{\omega_1 L_2^4}{16}\right)$$

Hence R_D can now be evaluated from Equation (2.11).

By substituting the value of R_D into Equation (2.10) the value of M_{AB} is computed.

The other values are written as

$$R_A = w_1(L_2 + L_1) + \omega_2 L_3 - R_D \tag{2.12}$$

$$M_{BD} = R_D L_2 - \frac{\omega_1 L_2^2}{2} \quad \text{and} \quad M_{BA} = M_{AB} \tag{2.13}$$

is already calculated

$$\text{Torque } T = \frac{M_{BA} + \left\{R_D L_2 - \frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2}\right\}}{2} \tag{2.14}$$

from statics the values of H_D can be evaluated.

Gould (1963) has introduced the effects of a vertical flexible support at D.

Case II (Case B of Gould): vertical reaction when the support D *is flexible*

Here Gould provides a flexible support at D. Let K be the stiffness of the spring. Case I is now modified.

$$K_v = \frac{R_D}{\delta R_D} \tag{2.15}$$

$$\frac{\partial U}{\partial R_D} = -\frac{R_D}{K_v} \tag{2.16}$$

hence

$$-\frac{R_D}{K_v}EI = C\left[L_2^3\left(\frac{\omega_1 L_2}{16} + \frac{R_D}{3}\right) + L_1^3\left(\frac{-5\omega_1 + 4}{12}\right) + \frac{L_1^2}{2}(\omega_1 + \omega_2 + 2L_2) + L_1\left(\frac{\omega_2 L_3^2}{2} + L_2^2\right)\right] = 0 \tag{2.17}$$

where C – intergrating factor

$$C = C_1 = \frac{L_1'}{L_2} \quad \text{or} \quad C = C_2 = \frac{L_2'}{L_1} \tag{2.17A}$$

R_D can now easily be calculated from Equation (2.17).

Case III (Case C of Gould): vertical and horizontal reactions on flexible supports at D

Again Gould suggests two flexible supports. A reference is made to Figure 2.1(e) to (g) and generalised equations are given by

$$\frac{\partial U}{\partial R_D} = \frac{-R_D}{K_V} \tag{2.18}$$

$$\frac{\partial U}{\partial H_D} = \frac{-H_D}{K_H} \tag{2.19}$$

$$M_{BD} = R_D L_2 - H_D H_2 \tag{2.20}$$

$$R_{BD} = R_D - \omega_1 L_2 - \frac{\omega_1 L_2^2}{2}$$

$$M_x = R_D x - \frac{H_2}{L_2} H_D x \frac{\omega_1 x^2}{2}$$

$$\frac{\partial M_x}{\partial R_D} = x \tag{2.21}$$

$$\frac{\partial M_x}{\partial H_D} = -\frac{H_2}{L_2} x$$

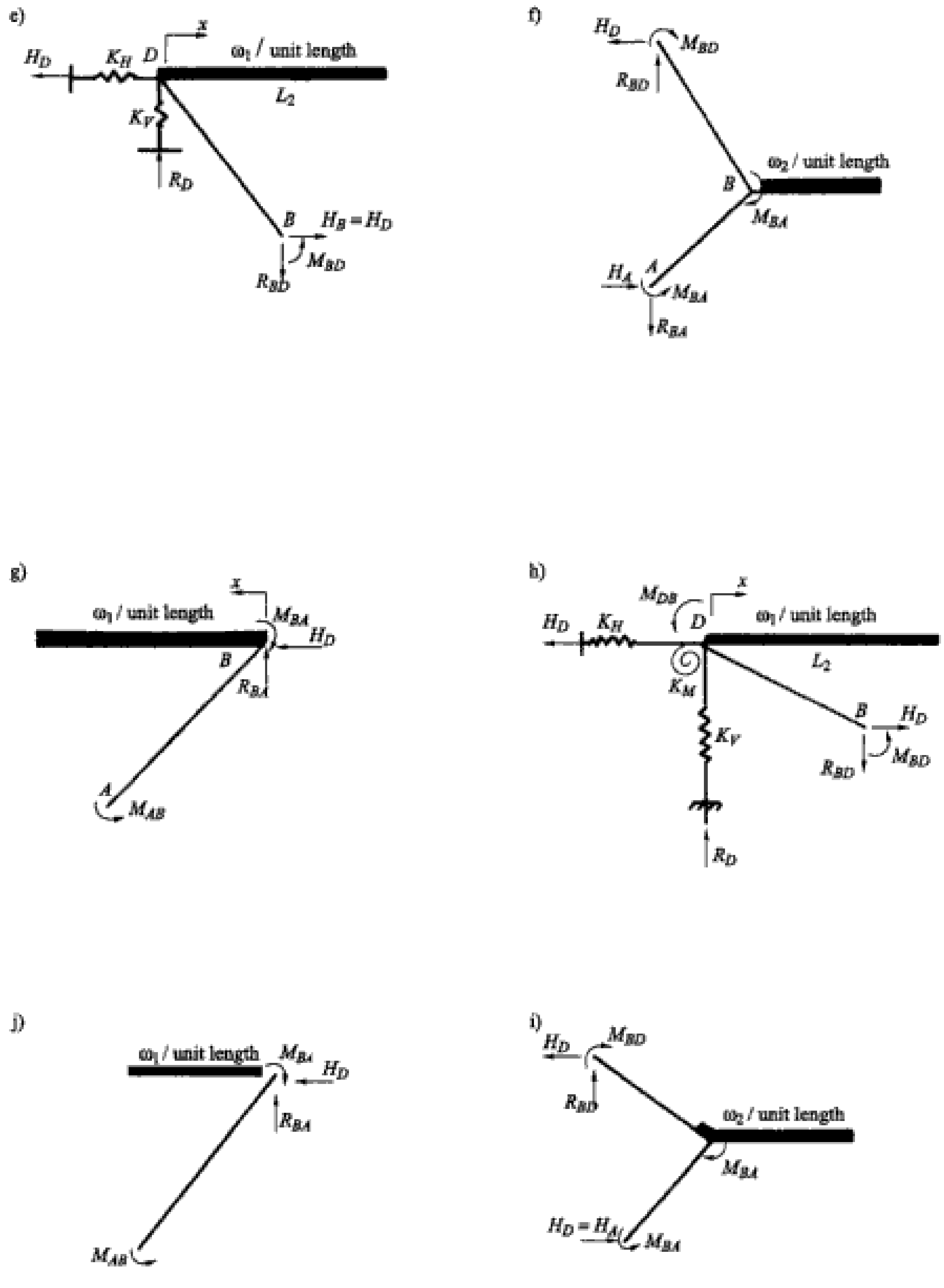


Figure 2.1 (cont.).

$$M_{BA} = R_D L_2 - H_D H_2 - \left\{ \frac{\omega_1 L_2^2}{2} - \frac{\omega_2 L_3^2}{2} \right\} \tag{2.22}$$

$$R_{BA} = R_D - \{\omega_1 L_2 + \omega_2 L_3\} \tag{2.23}$$

$$M_x = R_D L_2 - H_D H_2 - \left\{ \frac{\omega_1 L_2^2}{2} - \frac{\omega_2 L_3^2}{2} \right\} - \left\{ R_D - (\omega_1 L_2 + \omega_2 L_3) \right\} x - H_D \frac{H_1}{L_1} x + \omega_1 \frac{x^2}{2} \tag{2.24}$$

$$\frac{\partial M_x}{\partial R_D} = L_2 - x,$$

$$\frac{\partial M_x}{\partial H_D} = -H_2 - \left(\frac{H_1}{L} \right) x,$$

$$\frac{\partial M_x}{\partial M_{DB}} = -1 \tag{2.25}$$

Hence from Equations (2.18) and (2.19)

$$\frac{\partial U}{\partial R_D} = \frac{1}{EI} \int_0^{L_2} \left[R_D x - \left(H_2 - \frac{H_1}{L} x \right) H_D - \frac{1}{2} \frac{\omega_1 x^2}{2} \right] x \, \mathrm{d}x + \frac{1}{EI} \int_0^{L_1} \left[\frac{1}{2} \frac{\omega_1 x^2}{2} + (\omega_1 L_2 + \omega_2 L_3) x - \left(\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2} \right) + R_D (L_2 - x) + H_D \left(-H_2 \frac{H_1}{L_1} \right) \right] (L_2 - x) \, \mathrm{d}x = \frac{-R_D}{K_V} \tag{2.26}$$

$$\frac{\partial U}{\partial H_D} = \frac{1}{EI} \int_0^{L_2} \left[R_D x - (H_2 H_D x -) - \frac{1}{2} \frac{\omega_1 x^2}{2} (-H_2 x) \right] \mathrm{d}x + \frac{1}{EI} \int_0^{L_1} \left[\frac{1}{2} \frac{\omega_1 x^2}{2} + (\omega_1 L_2 + \omega_2 L_3) x - \left(\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2} \right) + R_D (L_2 - x) + H_D \left(-H_2 \frac{H_1}{L_1} \right) \right] (L_2 - x) \, \mathrm{d}x = \frac{-H_D}{K_H} \tag{2.27}$$

Case IV (Case D): vertical and horizontal reactions on flexible supports (support F *with partial fixity)*
It is assumed that the restraint to rotation is proportional to the angle of twist at support $D(M/\phi) = K_M$.

Case III is modified by adding the M_{DB} term to the moment expression. Again a generalised method is given, based on Gould (July 1963).

$$\frac{\partial U}{\partial M_D} = -\frac{M_{DB}}{K_M}$$

$$M_x = R_D x - \frac{H_2}{L_2} H_D x - \frac{\omega_1 x^2}{2} - M_{DB} \tag{2.27}$$

Figure 2.1(h)

a) $\dfrac{\partial M_x}{\partial R_D} = x$

b) $\dfrac{\partial M_x}{\partial H_D} = -H_2 x$

c) $\dfrac{\partial M_x}{\partial M_{DB}} = -1$

a) $M_{BD} = R_D L_2 - H_2 H_D - \dfrac{\omega_1 L_1^2}{2}$

b) $M_{BD} = R_D L_2 - H_2 H_D - \dfrac{\omega_1 L_1^2}{2}$ (2.28)

c) $M_{BC} = R_D L_2 - H_2 H_D - \left(\dfrac{\omega_1 L_1^2}{2} - \dfrac{\omega_2 L_3^2}{2}\right)$

Figure 2.1(i)

$$M_x = [R_D L_2] - H_2 H_D - M_{DB} - \left(\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_2^2}{2}\right)$$
$$- \left[R_D - (\omega_1 L_2 + \omega_2 L_3) - H_D \frac{\omega L_2}{2L_1} + \frac{\omega_1 x^2}{2}\right] \tag{2.29}$$

$$\frac{\partial M_x}{\partial R_D} = (L_2 - x), \quad \frac{\partial M_x}{\partial H_D} = -H_2 - \frac{H_1}{L_1}, \quad \frac{\partial M_x}{\partial M_{DB}} = -1 \tag{2.30}$$

Three equations are finally derived

$$\text{(A)}\quad \frac{\partial U}{\partial M_D} = -\frac{1}{EI}\int_0^{L_2}\left(R_D x - H_2 H_D x - \frac{1}{2}\frac{\omega_1 x^2}{2} - M_{DB}\right)(-1)\,\mathrm{d}x$$

$$+\frac{1}{EI}\int_0^{L_2}\left[\frac{\omega_1 x^2}{2} + (\omega_1 L_2 + \omega_2 L_3)x - \left(\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2}\right) + R_D(L_2 - x) + H_D\left(-H_2 - \frac{H_1}{L_1}x\right) - M_{DB}\right](-1)\,\mathrm{d}x$$

$$= -\frac{M_{DB}}{K_M} \tag{2.31}$$

or

$$\frac{1}{EI}[C_1 R_D + C_2 H_D + C_3\omega_1 + C_4\omega_2 + C_5 L_1 M_{DB}] = -\frac{M_{DB}}{R_M} \tag{2.32}$$

where

$$C_1 = 2(L_1 + L_2)^2,\quad C_2 = 2(-H_2 L_2^2 + 2H_2 L_1)H_D$$

$$C_3 = 2\left(-L_1^2 L_2 + \frac{L_2^2}{18}\right),\quad C_4 = 2(-L_3 - L_1 L_3^2),\quad C_5 = 2H_1 L_1 + L_1$$

$$\text{(B)}\quad \frac{\partial U}{\partial R_D} = -\frac{1}{EI}\int_0^{L_2}\left(R_D x - H_2 H_D x - \frac{\omega_1 x^2}{4} - M_{DB}\right)x\,\mathrm{d}x$$

$$+\frac{1}{E}\int_0^{L_2}\left[\frac{\omega_1 x^2}{4} + (\omega_1 L_2 + \omega_2 L_3)x - \left(\frac{\omega_1 L_1^2}{2} - \frac{\omega_2 L_3^2}{2}\right)\right]$$

$$+\left[R_D(L_2 - x) + H_D\left(-H_2 - \frac{H_1}{L_1}x\right) - M_{DB}\right](L_2 - x)\,\mathrm{d}x = -\frac{R_D}{K_V} \tag{2.34}$$

or

$$\frac{1}{EI}[C_6 R_D + C_7 H_D + C_8\omega_1 + C_9\omega_2 + C_{10} M_{DB}] = -\frac{R_D}{K_V} \tag{2.35}$$

where

$$C_6 = \frac{1}{3}(L_1^3 + L_2^3) - L_1 L_2(L_1 + L_2)$$

$$C_7 = -H_2L_2\left(L_1 + \frac{L_2^2}{3}\right) - \frac{H_1L_1L_2}{2}$$

$$C_8 = L_1^3\left(\frac{5}{6}L_1 + \frac{1}{12} - \frac{5}{6}L_2\right) + L_2^4 + \frac{L_1L_2^2}{2}$$

$$C_9 = \frac{1}{2}L_1L_2\left(L_1L_3 - L_3^2 - \frac{L_1L_2}{2}\right) - \frac{L_1^3L_3}{3} + \frac{1}{2}L_1^2\left(H_2 + \frac{H_1}{L_1}\right) \tag{2.36}$$

$$C_{10} = L_1\left(\frac{L_1}{2} - L_2\right)$$

(C) $$\frac{\partial U}{\partial H_D} = -\frac{1}{EI}\int_0^{L_2}\left(R_Dx - H_2H_Dx - \frac{\omega_1x^2}{4} - M_{DB}\right)(-H_2x)\,dx + \frac{1}{EI}\int_0^{L_2}\left[\frac{\omega_1x^2}{4} + (\omega_1L_2 + \omega_2L_3)x - \left(\frac{\omega_1L_1^2}{2} - \frac{\omega_2L_3^2}{2}\right)\right] + \left[R_D(L_2 - x) + H_D\left(-H_2 - \frac{H_1}{L_1}x\right) - M_{DB}\right] \times (-H_2 - \frac{H_1}{L_1}x)\,dx = -\frac{H_{DB}}{K_H} \tag{2.37}$$

or

$$\frac{1}{EI}[C_{11}R_D + C_{12}H_D + C_{13}\omega_1 + C_{14}\omega_2 + C_{15}M_{DB}] = -\frac{H_{DB}}{K_H} \tag{2.38}$$

where,

$$C_{11} = -H_2\left(\frac{L_1^3}{3} + \frac{L_1^2}{2}\right) - \frac{H_1}{L_1}\left(\frac{1}{2} + \frac{L_2}{3}\right)L_2^3$$

$$C_{12} = H_2^2\left(L_2 + \frac{L_1^3}{3}\right) + \frac{H_1L_2^2}{L_1}\left(H_2 - \frac{H_1L_2}{3L_1}\right)$$

$$C_{13} = \frac{H_2L_1^4}{16} - \frac{7H_2L_2^3}{12} - \frac{19}{48}\frac{H_1L_2^4}{L_1} + \frac{L_1L_2}{2}\left(H_2L_1 + \frac{H_1L_2}{4}\right) \tag{2.39}$$

$$C_{14} = H_2L_2L_3\left(\frac{L_2 + L_3^2}{2}\right) - \frac{L_2L_3H_1}{L_1}\left(\frac{L_2^2}{3} + \frac{L_2L_3}{2}\right)$$

$$C_{15} = \frac{1}{2}\left(H_2L_1^2 + \frac{H_1L_2^2}{2} + 2L_2\right)$$

EXAMPLE 2.1
Calculate reactions and moments for above cases using the following data for the cantilever staircase:
Stairs:
$L_1 = 1.78$, $H_1 = 1.50$ m
$L_2 = 3.048$ m, $H_2 = 2.45$ m
$L_3 = 1.22$ m
$\omega_1 = 14.36$ kN/m
$\omega_2 = 23.64$ kN/m on horizontal projection
$E = 2.07 \times 10^6$ kN/m

Dimensions:
$b = 305$ mm
$L = 6$ m
$d = 457$ mm
$G = 6.985$ kN/m^2
$I_{bx} = 1078 \times 10^6$ mm^4
$K = 48EI/L^3$
$K_H = 4728$ kN/m
$K_V = 10{,}683$ kN/m
total km/radian = 123,757 kN/m
total 2 No of $^1/_2$ length of beam = 61,879 kN/m
half length = 3 m
T = torsion = $\tau\beta d b^3 G$
B = width of stairs = 1.22 m
t = thickness of stairs = 165 mm = D_f

Table 2.1. Solution for case studies of a cantilever staircase.

	Case I	Case II	Case III	Case IV
R_D kN	5.39	5.65	28.25	29.31
R_A kN	92.65	93.00	70.40	69.35
M_{AB} kN m	110.00	110.40	51.53	50.00
M_{BD} kN m	49.20	50.00	35.12	34.45
M_{BA} kN m	35.30	32.41	17.50	16.81
T kN m	42.25	41.20	26.31	25.63
H_D kN	–	–	22.00	23.33

Note: $T = (M_{BD} + M_{BA})/2$.

2.5 A GENERALISED ANALYSIS OF STAIRS WITH UNSUPPORTED INTERMEDIATE LANDING

2.5.1 *Taleb's method (September 1964)*

This method is based on the principle of least work using equations of equilibrium of the entire stair and hence obtaining expressions directly for all redundants acting at the supports. In the plane of the flights shear, tension and compression are ignored. The load cases include symmetrically and unsymmetrically placed loads.

Notation for the analysis

L_1, H_1	=	horizontal and vertical projections of flight length, respectively;
L_3, B'	=	width and breadth of landing, respectively;
D_f	=	thickness of stair slab (waist);
L, B_1	=	length and width of flight, respectively;
α	=	angle of inclination of flight;
H_A, R_A, R_1	=	reactions at lower and upper supports corresponding to first system of H_D, R_D, R_2 coordinates;
M_x, M_y, M_z	=	moments in the direction of the x, y and z axes, respectively, corresponding to the first system of coordinates (Fig. 2.2);
M_{x1}, M_{y1}, M_{z1}	=	moments at upper and lower supports, whose vectors are parallel to the x, y and z;
M_{x2}, M_{y2}, M_{z2}	=	axes of the first system of coordinates;
N_1, Q_1, N_2, Q_2	=	reactions at upper and lower supports corresponding to the second system of coordinates (Fig. 2.3);
M_{X1}, M_{Y1}, M_{Z1} M_{X2}, M_{Y2}, M_{Z2}	=	moments at upper and lower supports, whose vectors are parallel to the X, Y and Z axes of the second system of coordinates;
P_1, P_4	=	unit dead loads of flight and landing, respectively;
P_2, P_3, P_5	=	unit imposed loads on lower flight, upper flight and landing, respectively;
E, G	=	moduli of elasticity and rigidity, respectively;
I_1, I_2	=	moments of inertia abut 1-1 and 2-2 axes, respectively;
I_p	=	polar moment of inertia;
$K_1 \ldots K_{15}$	=	constants;
P	=	$P_{(1+2)} + P_{(1+3)}$.

A reference is made to Figure 2.2, the equilibrium of the entire stair can be categorised as

$$\begin{aligned}
&\text{a)}\quad \sum F_X = 0 = \sum F_Y = 0 = \sum F_Z \\
&\text{b)}\quad R_1 = R_2 \\
&\text{c)}\quad H_D = H_A; \quad R_D = -R_A + B_1 P L_1 + L_3 B' P_{(4+5)}
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
&\text{a)}\quad \sum M_X = 0 = \sum M_Y = 0 = \sum M_Z \\
&\text{b)}\quad M_{X2} = M_{X1} - 2H_1 H_A + \frac{1}{2} B_1 P L_1^2 + L_3 B' P_{(4+5)} \\
&\text{c)}\quad M_{Y2} = M_{Y1} + f\Big(-R_A + B_1 L_1 P_{(1+2)} + \frac{1}{2} H_1 B' P_{(4+5)}\Big) \\
&\qquad M_{Z2} = M_{Z1} + f H_A
\end{aligned} \tag{2.41}$$

In this case L_1 and H_1 are the same as L_2 and H_2, respectively. Where L_1 and H_1 are different from L_2 and H_2, respectively, a reference is made to case studies in Section 2.4. The same is true if supports are flexible.

The strain energy in AB and DE stairs are now computed.

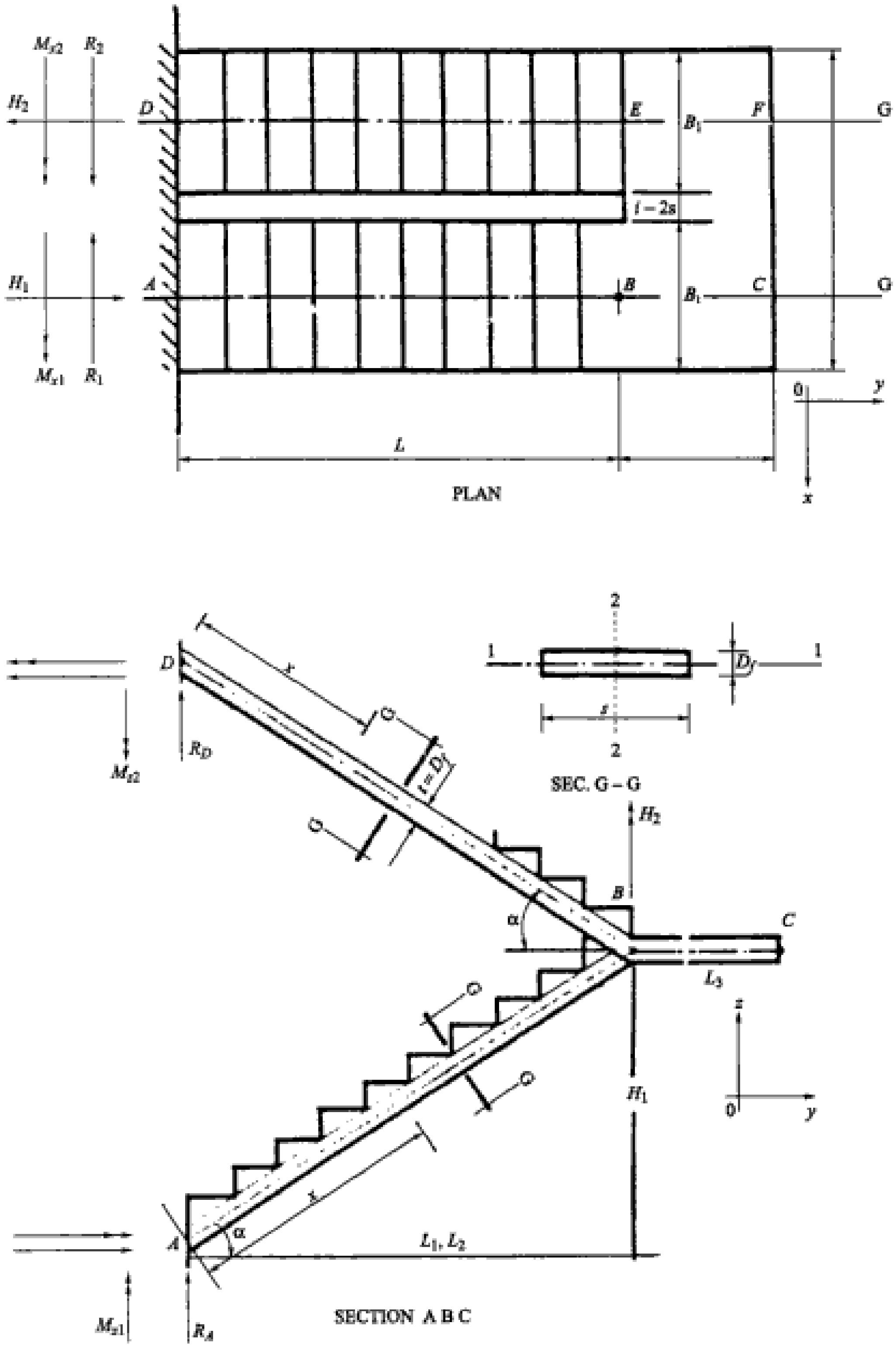


Figure 2.2. Stress resultants in y-z coordinates (Taleb 1964).

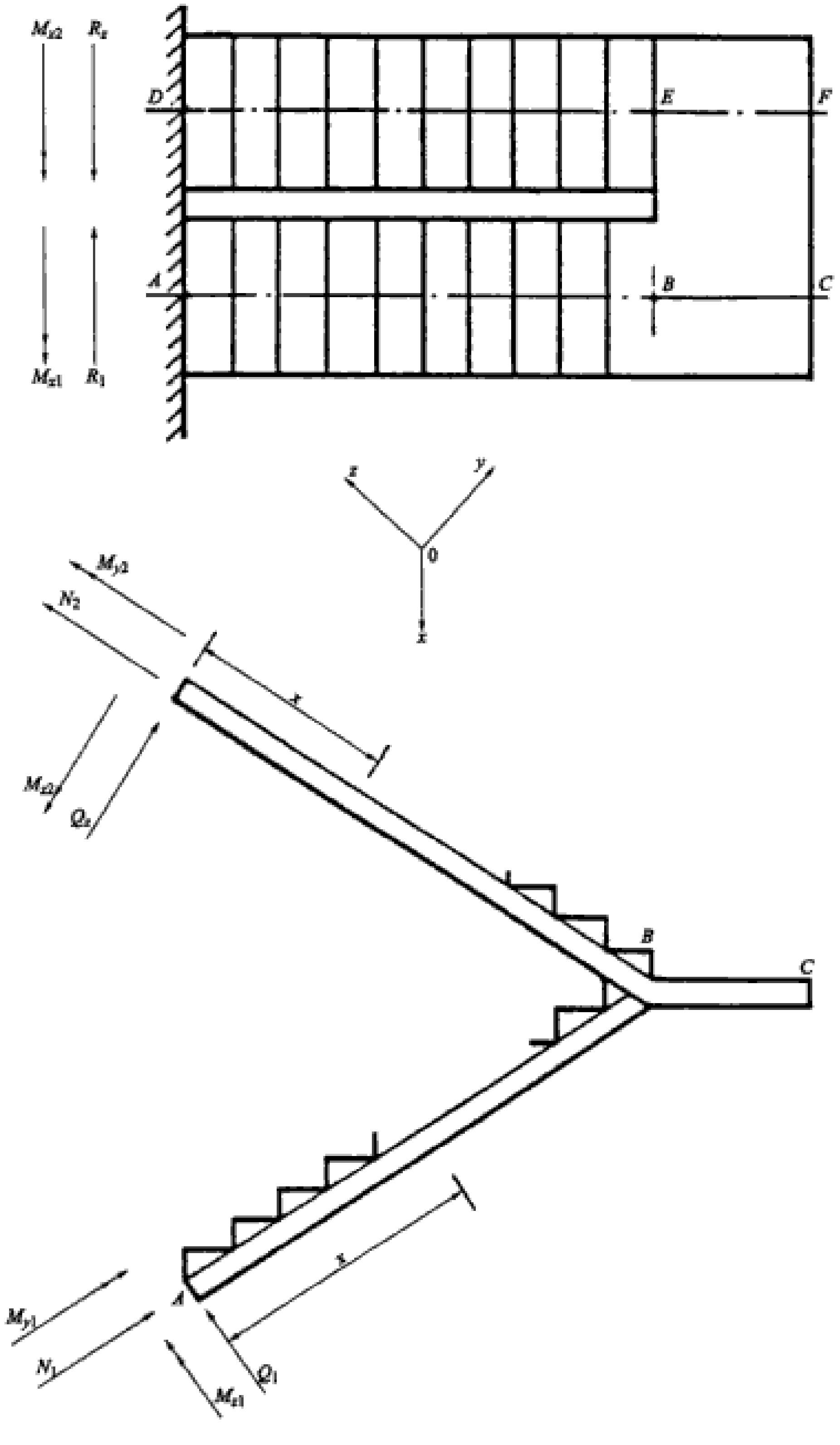


Figure 2.3. Stress resultants in x, y, z coordinates.

Let $U_{(AB)}$ be the strain energy in staircase AB.
Then

$$U_{(AB)} = \int_0^L \frac{M_{XAB}^2}{2EI_1}\,dx + \int_0^L \frac{(M_{Z1}\cos\alpha - M_{Y1}\sin\alpha - R_1 x)^2}{2EI_2}\,dx + \int_0^L \frac{(M_{Z1}\sin\alpha - M_{Y1}\cos\alpha)^2}{2GI_P}\,dx \tag{2.42}$$

where,

$$GI_P = \frac{2EI_1I_2}{I_1 + I_2} \tag{2.43}$$

and

$$M_{(XAB)} = M_{X1} + \frac{1}{2}B_1P_{(1+2)}x^2\cos^2\alpha + H_Ax\sin\alpha - R_Ax\cos\alpha$$

Let U_{DE} be the strain energy in staircase DE.
Similarly

$$U_{(AB)} = \int_0^L \frac{M_{XDE}^2}{2EI_1}\,dx + \int_0^L \frac{(M_{Z2}\cos\alpha - M_{Y2}\sin\alpha - R_1 x)^2}{2EI_2}\,dx + \int_0^L \frac{(M_{Z2}\sin\alpha - M_{Y2}\cos\alpha)^2}{2GI_P}\,dx \tag{2.44}$$

where,

$$M_{(XDE)} = M_{X2} + \frac{1}{2}B_1P_{(1+3)}x^2\cos^2\alpha + H_Dx\sin\alpha - R_Dx\cos\alpha \tag{2.45}$$

Total strain energy of the entire staircase:

$$U = U_{AB} + U_{DE} \tag{2.46}$$

Equations (2.40) and (2.41) indicate relevant moments and reactions at the support and may be expressed in terms of those at A which are taken as redundants. Six equations are written for the least work when no deflections or rotations at supports occur.

$$\frac{\partial U}{\partial H_1} = 0, \quad \frac{\partial U}{\partial R_A} = 0, \quad \frac{\partial U}{\partial R_2} = 0 \tag{2.47}$$

For example,

$$\frac{\partial U}{\partial M_{X1}} = 0, \quad \frac{\partial U}{\partial M_{Y1}} = 0, \quad \frac{\partial U}{\partial M_{Z1}} = 0$$

$$\frac{\partial U}{\partial x_1} = \int_0^L \frac{M^2_{(XAB)}}{EI_1}(x \sin\alpha)\,\mathrm{d}x + \int_0^L \frac{(M_{(XDE)})}{EI_1} x(-2H_1 + x\sin\alpha)\,\mathrm{d}x$$

$$+ \int_0^L \frac{M_{Z2}\cos\alpha}{EI_2} + (M_{Y2}\sin\alpha - R_1 x)(f\cos\alpha)\,\mathrm{d}x$$

$$+ \int_0^L \left(\frac{M_{Z2}\sin\alpha - M_{Z2}\cos\alpha}{GI_P}\right)(f\sin\alpha)\,\mathrm{d}x = 0 \qquad (2.48)$$

The above equations are solved for the unknowns H_A, R_A, R_1, M_{X1}, M_{Y1}, and M_{Z1}.

The solutions are given below for unsymmetrical loading:

$$R_A = -\frac{K_{15}}{K_9} \qquad (2.49)$$

$$H_A = -\frac{K_{14}}{K_4} \qquad (2.50)$$

$$R_1 = -3K_7\left(K_{10} + \frac{fK_{15}}{K_9}\right) = -3K_7(K_{10} - fR_A) \qquad (2.51)$$

$$M_{X1} = \frac{1}{2}\left(\frac{2L_1K_{14}}{K_4} - \frac{L_1K_{15}}{K_9} + K_{13}\right)$$

$$= \frac{1}{2}(-H_1H_A + L_1R_A + K_{13}) \qquad (2.52)$$

$$M_{Y1} = -\frac{1}{2}\left(-\frac{fK_3K_{14}}{K_2K_4} + \frac{fK_{15}}{K_9} + K_{10}\right)$$

$$= -\frac{1}{2}\left(-\frac{fK_3H_A}{K_2} + fR_A + K_{10}\right) \qquad (2.53)$$

$$M_{Z1} = \frac{fK_{14}}{2K_4} - K_8K_{15} - \frac{H_1K_{10}}{2L_1} - \frac{2L^2K_7K_{10}}{L_1}$$

$$= \frac{1}{2}fH_A + K_8R_A - \frac{H_1K_{10}}{2L_2L^2K_7K_{10}/L_1} \qquad (2.54)$$

where,

$$\overline{m} = \frac{I_1}{I_2}, \quad \bar{n} = \frac{EI_1}{GI_P} = \frac{1}{2}(\overline{m} + I), \quad f = B - B_1$$

and

$$K_1 = \overline{m}\cos^2\alpha + \bar{n}\sin^2\alpha, \quad K_2 = \overline{m}\sin^2\alpha + \bar{n}\cos^2\alpha$$

$$K_3 = (\overline{m} - \bar{n})\sin^2\alpha, \quad K_4 = \frac{2H_1^2}{3} + \frac{1}{2}f^2\left(K_1 - \frac{K_3^2}{K_2}\right)$$

$$K_5 = 4K_1L^2 - 3\overline{m}L_1^2, \quad K_6 = 4K_3L^2 - 3\overline{m}L_1H_1;$$

$$K_7 = \frac{L_1K_3 - H_1K_1}{K_5}, \quad K_8 = \frac{f}{2L_1}(H_1 + 4L^2K_7)$$

$$K_9 = \frac{L_1^2}{6} + \frac{f^2}{2L_1(L_1K_2 - H_1K_3 - K_6K_7)}$$

$$K_{10} = f\left(B_1P_{(1+2)}L_1 + \frac{1}{2}L_3B'P_{(4+5)}\right)$$

$$K_{11} = \frac{B_1L_1^2H_1}{24(P_{(1+2)} - 7P_{(1+3)})} - L_3B'P_{(4+5)}H_1\left(\frac{5L_1}{6} + \frac{3H_1}{4}\right)$$

$$K_{12} = \frac{B_1L_1^3}{8}\left(P_{(1+3)} - P_{(1+2)} - \frac{B_1PL_1^3}{12} + L_3B'P_{(4+5)}L_1\right) \times \left(\frac{L_1}{6} + \frac{L_3}{4}\right) - fK_2K_{10}$$

$$K_{13} = \frac{B_1L_1^2}{6}\left(P_{(1+3)} - P_{(1+2)} + \frac{1}{2}L_3B'P_{(4+5)}\right)(L_1 + L_3)$$

$$K_{14} = K_{11} + H_1K_{13}$$

$$K_{15} = K_{12}\frac{1}{2L_1K_{13}} + \frac{fK_{10}}{2L_1}(L_1K_2 + H_1K_3 + K_6K_7)$$

$$N_1 = R_A\sin\alpha + H_A\cos\alpha Q_1 = R_A\cos\alpha + H_A\sin\alpha \tag{2.55}$$

$$M_{Y1} = M_{y1}\cos\alpha + M_{z1}\sin\alpha, \quad M_{Z1} = M_{z1}\cos\alpha + M_{y1}\sin\alpha \tag{2.56}$$

$$M_{X1} = M_{x1} \tag{2.57}$$

and

$$N_2 = H_D\cos\alpha + R_D\sin\alpha, \quad Q_2 = R_D\cos\alpha - H_2\sin\alpha \tag{2.58}$$

$$M_{Y2} = M_{y2}\cos\alpha + M_{z2}\sin\alpha \tag{2.59}$$

$$M_{Z2} = M_{z2}\cos\alpha + M_{y2}\sin\alpha \tag{2.60}$$

$$M_{X2} = M_{x2} \tag{2.61}$$

A special case is made for Symmetrical Loading. The equations are produced when:

a) the flights only are loaded;
b) the landing is loaded;
c) both flights and landing are loaded.

$$P_2 = P_3 \quad \text{or} \quad P_{(1+2)} = P_{(1+3)}, \quad P = 2P_{(1+2)}$$

The above equations are resolved and the following expressions are obtained:

$$R_1 = R_2 = 0$$

$$R_A = R_D = B_1 L_1 P_{(1+2)} + \frac{1}{2} L_3 B' P_{(4+5)} \tag{2.62}$$

$$H_A = H_D = \frac{H_1}{K_4}\left[\frac{1}{4} B_1 L_1^2 P_{(1+2)} + L_3 B' P_{(4+5)}\left(\frac{1}{3} L_1 + \frac{1}{4} H_1\right)\right] \tag{2.62a}$$

$$M_{X1} = M_{X2} = \frac{1}{2}\left[\left(-2H_1 H_A + L_1 R_A + \frac{1}{2} L_3 B' P_{(4+5)}\right)(L_1 + H_1)\right]$$

$$M_{Y1} = M_{Y2} = -f\frac{K_3 H_A}{2K_2}$$

$$M_{Z1} = M_{Z2} = \frac{1}{2} f H_A$$

Note: where flexible supports are included at D, the spring constants of K_H, K_v and K_M are simulated in the above equations thus modifying H_D, M_{DB} etc. The whole equation on the lines suggested in Section 2.4 can be rewritten and then finally solved for various unknowns.

EXAMPLE 2.2

Analyse a staircase with the following dimensions and parameters for loading placed unsymmetrically and symmetrically:

$H_1 = 1.83$ m,	$L_1 = 1.22$ m,	$B_1 = 1.22$ m,	$B' = 2.74$ m
$D_f = 140$ mm,	$m = 0.013$,	$n = 0.507$	
$f = 1.52$ m,	$\cos\alpha = 0.848$,	$\sin\alpha = 0.533$;	$L = 3.4$ m

Loads: $P_1 = P_4 = 4.8\ \text{kN/m}^2$, $P_2 = P_5 = 4.8\ \text{kN/m}^2$, $P_3 = 0$

Unsymmetrical $P_{(1+2)} = 9.6\ \text{kN/m}^2$, $P_{(1+3)} = 4.8\ \text{kN/m}^2$

$P_{(4+5)} = 9.6\ \text{kN/m}^2$, $P = 1.4\ \text{kN/m}^2$

Symmetrical $P_1 = P_2 = P_3 = P_4 = P_5 = 4.8\ \text{kN/m}^2$

$P_{(1+2)} = P_{(1+3)} = P_{(4+5)} = 9.6\ \text{kN/m}^2$

SOLUTION

Staircase under unsymmetrical and symmetrical loads

General constants not dependent on loads:

$K_1 = 0.1535$, $K_2 = 0.3667$, $K_3 = -0.0162$, $K_4 = 2.2514$

$$K_5 = 4 \times 0.1535 \times 3.4^2 - 3 \times 0.013 \times 3^2 = 7.0984 - 0.351 = 6.7474$$

$$K_6 = 4 \times 0.1535 \times 3.4^2 - 3 \times 0.013 \times 3 \times 1.83 = 7.09784 - 0.21411$$
$$= 6.8837$$

$$K_7 = \frac{3 \times (-0.0162) - 1.83 \times 0.1535}{6.7474} = \frac{-0.0486 - 0.2809}{6.7474} = 0.0488$$

$$K_8 = \frac{1.52}{2 \times 3}[1.83 + 4 \times 3.4^2 \times (-0.0488)] = \frac{1.52}{6}(1.83 - 2.256512)$$
$$= 0.4636$$

$$K_9 = \frac{3^2}{6} + \frac{1.52^2}{2 \times 3}[3 \times 0.3667 - 1.83 \times (-0.0162) - 6.8837(-0.0488)]$$
$$= 1.5 + 0.3850667[1.100 + 0.029646 + 0.3359246] = 2.0644$$

Unsymmetrical placed loads:

$$K_{10} = 1.52(1.22 \times 9.6 \times 3 + \frac{1}{2}1.22 \times 2.74 \times 9.6)$$
$$= 1.52 \times (35.136 + 16.04544) = 77.7958$$

$$K_{11} = \left(1.22 \times 3^2 \times \frac{1.83}{24}\right)(9.6 - 7 \times 4.8) - 1.22 \times 2.74 \times 9.6$$
$$\times 1.83\left(5 \times \frac{3}{6} + 3 \times \frac{1.83}{4}\right) = 247.5110$$

$$K_{12} = \left[[1.22 \times \left(\frac{3}{8}\right)^3\right](4.8 - 9.6) - 1.22 \times 4.8 \times \frac{3^3}{12} + 1.22 \times 2.74 \times 9.6$$
$$\times 3\left(\frac{3}{6} + \frac{1.22}{4}\right) - 1.52 \times 0.3667 \times 77.7958 = 1.1973$$

$$K_{13} = 1.22 \times \frac{3^2}{6}(4.8 - 9.6) + \frac{1}{2} \times 1.22 \times 2.74 \times 9.6 \times 4.22$$
$$= 41.3595$$

$$K_{14} = -247.5110 + 1.83 \times 41.3598 = 171.8226$$

$$K_{15} = 1 - 1.973 \times \frac{1}{2} \times 3 \times 41.3595 + \left(1.52 \times \frac{77.7958}{6}\right)$$
$$\times \left[3(0.3667) + 1.83(-0.0162) + 6.8837(-0.0488)\right]$$
$$= -62.03925 + 15.673603 = -46.3656$$

$$N_1 = \frac{46.3656}{2.0644} \times 0.533 + \frac{171.8226}{2.2514} \times 0.848 = 76.6888$$
$$Q_1 = \frac{46.3656}{2.0644} \times 0.848 - \frac{171.8226}{2.2514} \times 0.533 = -21.6319$$

Substituting into Equations (2.49) to (2.54)

$$R_A = 22.4596, \quad H_A = 76.3181 \text{ units kN}$$
$$R_1 = -3(-0.0488)(77.7958 - 1.52 \times 22.4596) = 6.3914 \text{ kN}$$
$$M_{x1} = \frac{1}{2}(-2 \times 1.83 \times 76.3181 + 3 \times 22.4596 + 41.3598)$$
$$= -85.29281 \text{ kN m}$$
$$M_{y1} = -\frac{1}{2}(1.52 \times 41.3598 \times \frac{76.3181}{0.3667} - 1.52 \times 22.4596 + 77.7958$$
$$= -\frac{1}{2}\left(13083.943 - 34.138592 + 77.7958)\right) = -6563.8 \text{ kN m}$$
$$M_{z1} = -\frac{1}{2} \times 1.52 \times 76.3181 + (-0.4636)\frac{22.4596}{6}$$
$$- 1.83 \times \frac{77.7958}{6} - 2 \times 3.4^2(-0.0488)\frac{77.7958}{3}$$
$$= -58.001756 - 1.7353784 - 23.727719 + 29.257859$$
$$= 54.2070 \text{ kN m}$$
$$M_{Y1} = -6563.8 \times 0.848 + 54.2070 \times 0.533 = -5537.210 \text{ kN m}$$
$$M_{Z1} = 54.2070 \times 0.848 + 6563.8 \times 0.533 = 35.444729 \text{ kN m}$$
$$M_{X1} = M_{x1} = -85.2928 \text{ kN m}$$

$$M_{x2} = 6563.8 - 2 \times 1.83 \times 76.3181 \times \frac{1}{2} \times 1.22 \times 4.8 \times 3^2$$
$$+ 1.22 \times 2.74 \times 9.6\left(3 + \frac{1}{2} \times 1.22\right)$$
$$= 6563.8 - 279.32425 + 26.352 + 115.84808 = 6426.676 \text{ kN m}$$
$$M_{y2} = -6563.8 + 1.52\Big(22.4596 + 1.22 \times 3 \times 9.6$$
$$+ \frac{1}{2} \times 1.83 \times 2.74 \times 9.6\Big)$$
$$= -6563.8 + 1.52(-22.4596 + 35.136 + 24.06816)$$
$$= -6563.8 + 55.851731 = 6507.9483 \text{ kN m}$$
$$M_{z2} = M_{Z2} \times 0.848 - M_{Z2} \times 0.533$$
$$= 170.2105 \times 0.848 + 6507.9483 \times 0.533 = 3483.1214 \text{ kN m}$$

$$M_{Y2} = M_{y2} \times 0.848 - M_{z2} \times 0.533 = -6507.9483 \times 0.848$$
$$- 170.2105 \times 0.533 = -5609.4624 \text{ kN m}$$
$$M_{Z2} = M_{z2} \times 0.848 - M_{y2} \times 0.533 = 170.2105 \times 0.848$$
$$+ 6507.9483 \times 0.533 = 3483.1214 \text{ kN m}$$
$$M_{X2} = M_{x2} = 6426.676 \text{ kN m}$$

$$R_D = -22.4596 + 1.22 \times 4.8 \times 3 + 1.22 \times 2.74 \times 9.6$$
$$= -22.4596 + 17.568 + 32.09088 = 27.19928 \approx 27.2 \text{ kN}$$

$$H_D = H_A = 76.3181 \text{ kN} \approx 76.32 \text{ kN}$$

2.6 METHOD OF SPACE INTERSECTIONS OF PLATES

2.6.1 *Liebenberg method (May 1960)*

This method has been developed using the extensional (membrane or planar) stiffness produced by the interaction of the stair flights and landings. The landings, columns, walls, beams and floor slabs are 'points' or 'lines' of intersection and they are treated as support to the 'secondary load carrying system' of the bending stresses in the slab elements. The extensional or membrane forces at the intersection points provide reactions which balance the shear forces in the slab elements.

The analytical procedure is summarised below and in Table 2.2.

a) First the conditions for a proper function of the primary system must exist. The effective supports are provided to the secondary bending system and local direct forces due to applied loading when inclined to the axis of the stair.

b) A provision of imaginary external restraints at supports thus preventing displacements and rotations.

c) Calculations of the resultant reactions acting on the imaginary supports.

d) Determination of the magnitude of the forces in the primary system and the actual stresses due to the combined effect of the forces on both primary and secondary systems.

Notation for the analysis

RB_{CD} = the resultant reaction due to the secondary bending force system and local direct forces acting at the intersection line CD at an angle α with the vertical;

R_{FCD} = the reaction at CD due to the bending forces in the flight acting at right angles to the flight;

R_{LCD} = the reaction at CD due to the bending forces in the landing acting at right angles to the landing;

D_{FCD} = the reaction at CD due to the local direct forces in the flight acting in the plane of the flight;

RE_{FCD} = the resultant extensional force in the flight at CD due to the reaction RB_{CD} but not including the effect on the local direct forces in the flight;

RE'_{FCD} = the resultant extensional force in the flight at CD due to the reaction RB_{CD} and including the effect of the local direct forces in the flight;

RE_{LCD} = the resultant extensional force in the landing at CD;

E_{EC} = the extensional force per unit length in the flight at C not including the local direct force;

E'_{FC} = the extensional force per unit length in the flight at C including the local direct force;

E_{LC} = the extensional force per unit length in the landing at C;

V_{CD} = the shear force acting along the intersection line CD due to the primary force system but not including the effect of local direct forces;

V'_{CD} = the shear force acting along the intersection line CD due to the primary force system including the effect of local direct forces

Table 2.2. Analysis of stairs.

Case I: stairs as a triangular arch

The stair is considered as a triangular arch. An applied line or knife edge loading is acting at the steps (Figs 2.4(a) to (h)). The top or bottom slabs are restrained against horizontal movements.

Case II: cantilever landing slabs with single flights

A reference is made to Figures 2.4(a) and (b). A cantilever landing slab with a single flight rigidly supported at the lower end.

$$RE'_{FCD} = RE_{FCD} + D_{FCD} \tag{a}$$

where, D_{FCD} is the local extension or membrane force and

$$RE_{FCD} = -R_{BCD}\frac{\cos\beta}{\sin\alpha} \tag{b}$$

$$RE_{LCD} = \text{resultant extension or membrane force} = -R_{BCD}(\sin\beta + \cos\beta\cdot\cot\alpha) \tag{c}$$

Case I
Knife edge loading ω/unit length

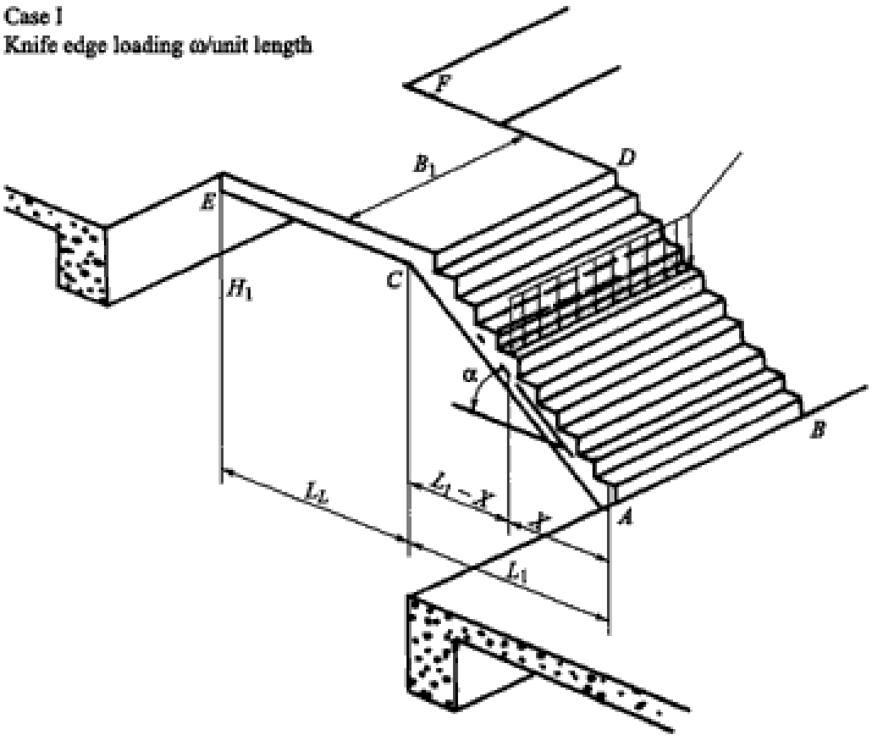


Figure 2.4(a). Staircase with parameters.

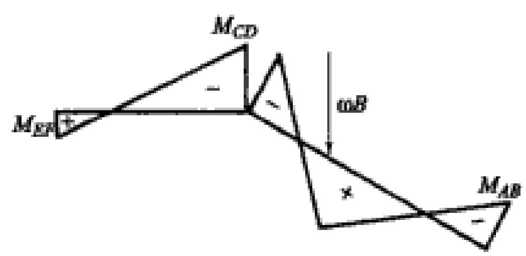


Figure 2.4(b). Bending moments.

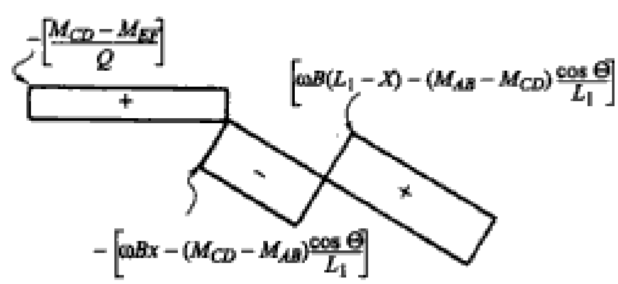


Figure 2.4(c). Shear forces.

Table 2.2 (cont.).

The force distribution is taken as linear, hence

$$E_{FC} = -\left[R_{BCD}\frac{\cos\beta}{\sin\alpha}\left(1+\frac{6e}{B}\right)\right] \quad \text{per unit width} \tag{d}$$

and

$$E'_{FC} = \left[R_{BCD}\frac{\cos\beta}{\sin\alpha}\left(1+\frac{6e}{B}\right)\right] + \left[\frac{D_{FCD}(1+6e')}{B^2}\right] \quad \text{per unit width} \tag{e}$$

where e' is the eccentricity of D_{FCD}

$$E_{FCD} = -\left[R_{BCD}\frac{\cos\beta}{B\sin\Theta}\left(1-\frac{6e}{B}\right)\right] \quad \text{per unit width} \tag{f}$$

$$E'_{FCD} = -\left[R_{BCD}\frac{\cos\beta}{B\sin\alpha}\left(1-\frac{6e}{B}\right)\right] + \left(\frac{D_{FCD}(1-6'')}{B^2}\right) \quad \text{per unit width} \tag{g}$$

$$E_{LC} = \frac{RE_{LCD}}{B}\left[1+\frac{6e}{B}\right] \quad \text{per unit width} \tag{h}$$

$$E_{LD} = \frac{RE_{LCD}}{B}\left[1-\frac{6e}{B}\right] \quad \text{per unit width} \tag{i}$$

Considering the equilibrium of the flight

$$RE_{FAB} = RE_{FCD} = R_{BCD}\frac{\cos\beta}{\sin\alpha} \tag{j}$$

$$RE'_{FAB} = RE'_{FCD} = -P_F\sin\Theta - R_{BCD}\frac{\cos\alpha}{\sin\alpha} + D_{FCD} - P_F\sin\alpha \tag{k}$$

(where P_F is the total load on the flight)

$$V_{AB} = V_{CD} = \frac{1}{\sqrt{(H_1^2+L_1^2)}}\left[-RE_{FCD}\frac{3}{2}e\right] = \frac{1}{\sqrt{(H_1^2+L_1^2)}}\left[R_{BCD}\frac{\cos\beta}{\sin\alpha}\frac{3}{2}e\right] \tag{l}$$

$$V'_{AB} = V'_{CD} = \frac{1}{\sqrt{(H_1^2+L_1^2)}}\left[R_{BCD}\frac{\cos\beta}{\sin\alpha}\frac{3}{2}e - D_{FCD}e' + P_F\sin\alpha e'' + D_{FAB}e'''\right] \tag{m}$$

where e'' is the eccentricity of the resultant of P_F and e''' is the eccentricity of the local direct force D_{FAB}

$$E_{FA} = \frac{RE_{FAB}}{B}\left(1-\frac{3e}{B}\right) \quad \text{per unit width} \tag{n}$$

$$E'_{FA} = -E_{FA} + \frac{D_{FAD}}{B}\left(1-\frac{6e'''}{B}\right) \quad \text{per unit width} \tag{o}$$

$$E_{Fb} = \frac{RE_{FAB}}{B}\left(1+\frac{3e}{B}\right) \quad \text{per unit width} \tag{p}$$

$$E'_{Fb} = E_{Fb} + \frac{D_{FAb}}{B}\left(1-\frac{6e'''}{B}\right) \quad \text{per unit width} \tag{q}$$

Considering the equilibrium of the landing:

$$V_{FE} = -RE_{LCD}$$

$$E_{LF} = \frac{V_{CD}}{L_L} = \frac{6}{L_L^2}\left[-RE_{LCD}\left(\frac{B}{2}+e+c\right) - V_{CD}\cdot\frac{L_L}{2}\right] \quad \text{per unit width} \tag{r}$$

$$E_{LE} = \frac{V_{CD}}{L_L} = \frac{6}{L_L^2}\left[-RE_{LCD}\left(\frac{B}{2}+e+c\right) + V_{CD}\cdot\frac{L_L}{2}\right] \quad \text{per unit width} \tag{s}$$

Case I (cont.)

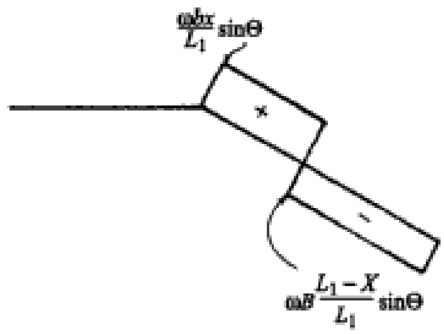


Figure 2.4(d). Local direct forces.

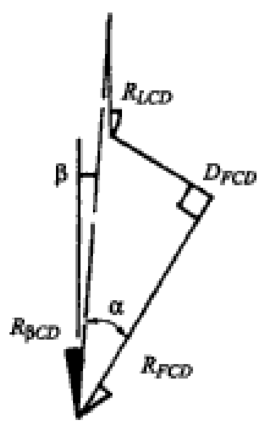


Figure 2.4(e). Force diagram.

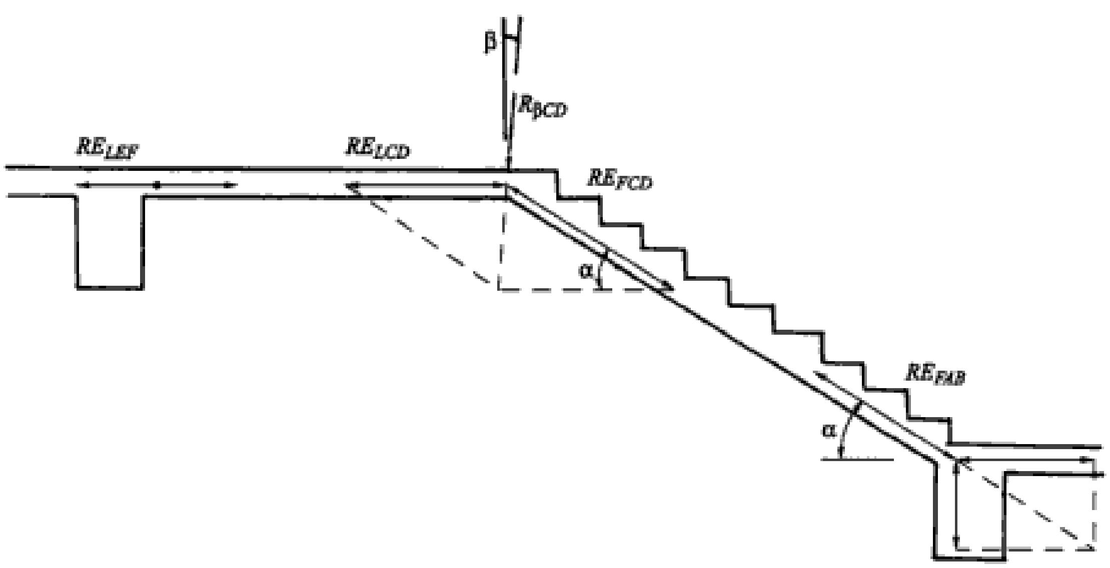


Figure 2.4(f). Resultant extensional forces.

Table 2.2 (cont.).

When one edge of the flight (BD) is built into a wall an additional resistance in the form of a shear force V_{BD} acts along the edge of the flight. Due to this additional support, the slab is treated as if it is lying in three supports when bending forces are evaluated. The equilibrium of the flight gives:

$$RE'_{FAB} = RE'_{FCD} - P_F \sin\alpha + V'_{BD} \tag{u}$$

i.e.

$$V'_{BD} - RE'_{FAB} = -RE'_{FCD} + P_F \sin\alpha \tag{w}$$

i.e.

$$V'_{BD} = K_S\left(-RE'_{FAB} + P_F \sin\alpha\right) \tag{x}$$

and

$$-RE'_{FAB} = (1 - K_S)\left(-RE'_{FCD} + P_F \sin\alpha\right) \tag{y}$$

where K_S cannot be determined by simple methods.

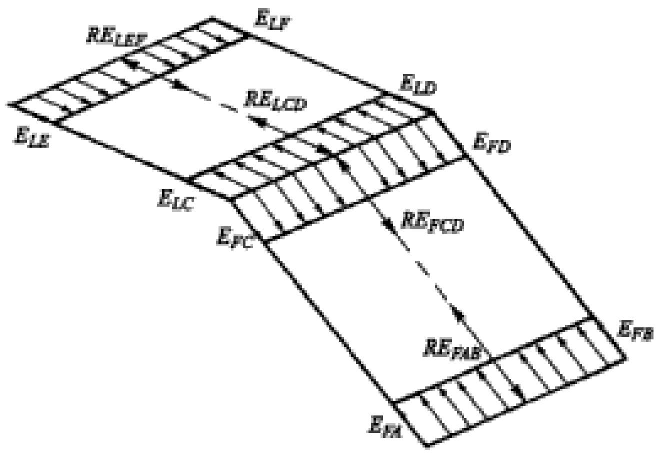


Figure 2.4(g). Extensional forces.

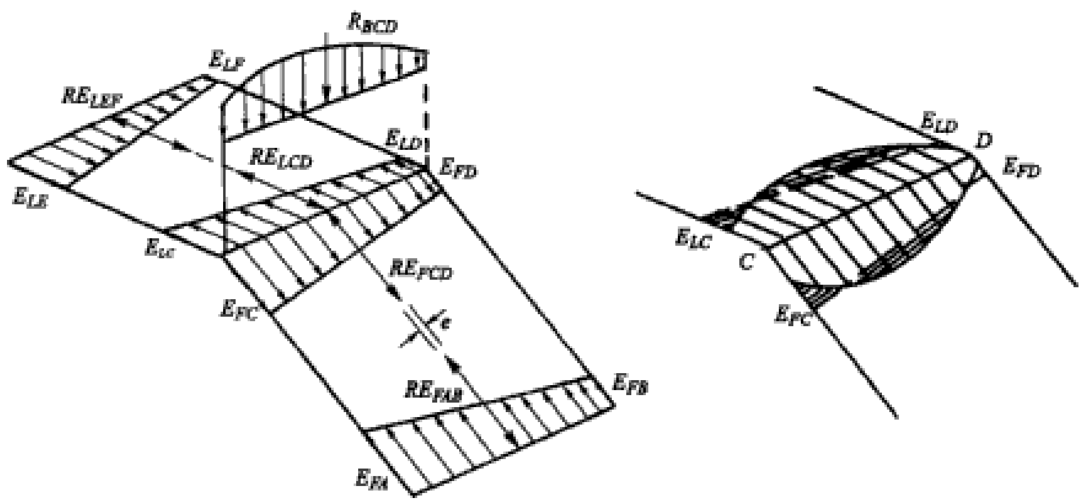


Figure 2.4(h). Reaction distribution.

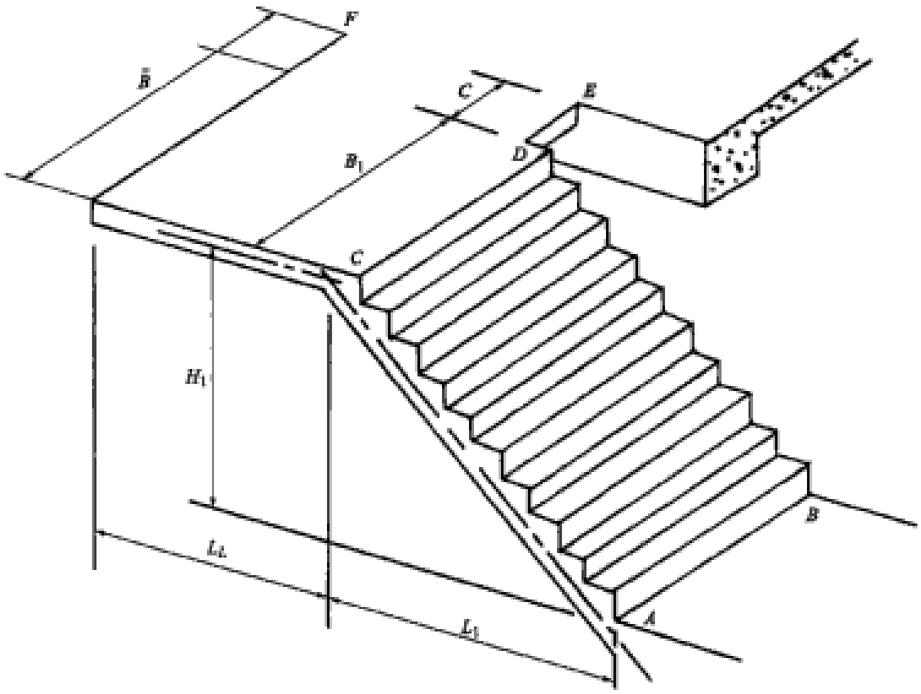


Figure 2.4 i(a). Cantilever landing slab with single flight.

Case II

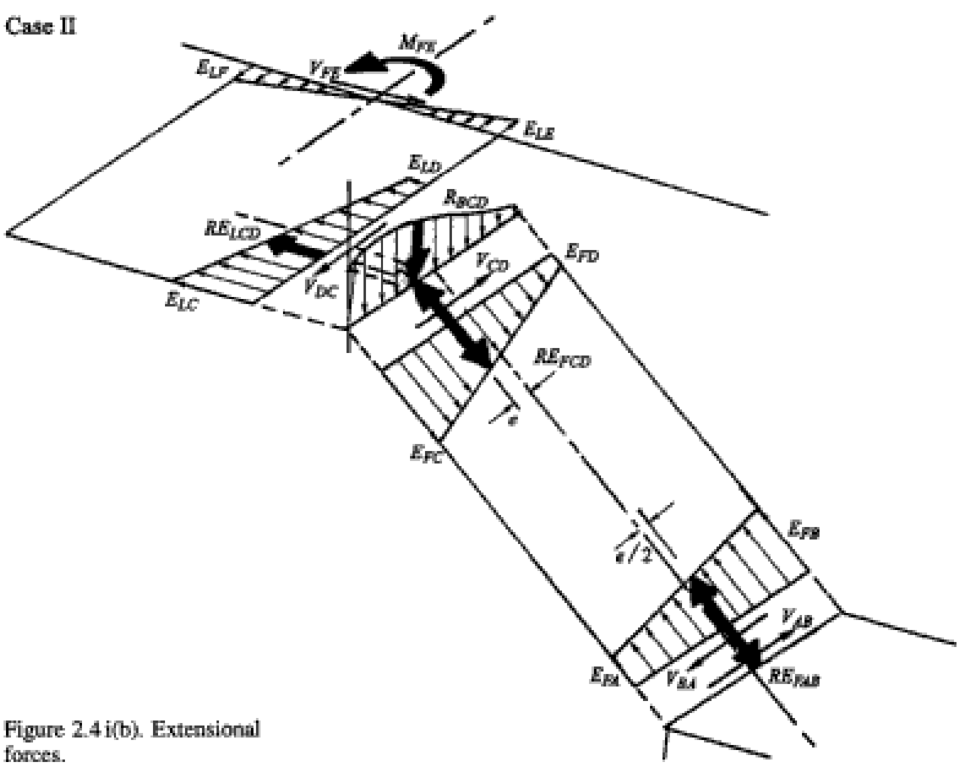


Figure 2.4 i(b). Extensional forces.

Table 2.2 (cont.).

Case III: scissors type staircases

Here two similar staircases are joined to form 'scissors' and are supported on the main landing as shown in Figures 2.4j(a) and j(b). It becomes a statically intermediate structure. The system requires an additional force 'H' acting as shown.

$$H = RE''_{FGH}\sin\Theta_2 - RE''_{FCD}\sin\Theta_1 - RE''_{FGH}\cos\Theta_2\cos\alpha_2 - (RE''_{FCD}\cos\Theta_1\cos\alpha_1) \tag{a}$$

$$= R_{BGH}\sin\beta \tag{b}$$

$$(RE''_{FGH}\cos\Theta_2\sin\alpha_2 - \left(RE''_{FCD}\cos\Theta_1\cdot\sin\alpha_1\right) = R_{BGH}\cos\beta \tag{c}$$

or

$$H = RE''_{FGH}(\sin\Theta_2 - \cos\Theta_2\cos\alpha_2) - RE''_{FCD}(\sin\Theta_1 + \cos\Theta_1\cos\alpha_1) = R_{BGH}\cos\beta$$

If bending of the landing dominates, the displacement of the resultant X_{AB} and X_{JK} will depend on the edges of the landings in the direction coinciding with the planes of the flights.

If the landings and flights are equal

$$X_{JK} = X_{AB} = \frac{K}{2} \tag{d}$$

If the upper landing has negligible stiffness K compared with the lower landing

$$X_{JK} = 0, \quad X_{AB} = K \tag{e}$$

Case IV: membrane type staircase

The extensional or membrane forces of a uniformly distributed load can be calculated in various planes with B_1 and B_2 values small (Figs 2.4k(a) to k(d)). ω_1 and ω_2 should be the uniform load per unit horizontal area of the stairs. The following computational values can be obtained in a generalised manner:

$$m_2 = m_1 = \left[k_1\omega_1 L_1 B + k_2\omega_2 L_L\left(B + \frac{C}{2}\right)\right]\frac{1}{B} = m \tag{a}$$

$$RE_{FCD} = -RE_{FGH} = -\frac{mB}{\sin\alpha} \tag{b}$$

$$RE'_{FCD} = -RE'_{FGH} = -\frac{mB}{\sin\alpha} + \frac{\omega_1 L_1 B\sin\alpha}{2} \tag{c}$$

The magnitude of the extensional forces are as follows:

$$E_{FA} = \frac{RE_{FCD}}{B}\left[1 - \frac{3(B+C)}{B}\right] \tag{d}$$

$$= -\frac{m}{\sin\alpha}\left[1 - \frac{3(B+C)}{B}\right] \quad \text{per unit width} \tag{e}$$

$$E'_{FA} = E_{FA} - \omega_1 L_1\frac{\sin\alpha}{2} \quad \text{per unit width} \tag{f}$$

$$E_{Fb} = \frac{RE_{FCD}}{B}\left[1 + \frac{3(B+C)}{B}\right]$$

$$= -\frac{m}{\sin\alpha}\left[1 + \frac{3(B+C)}{B}\right] \quad \text{per unit width} \tag{g}$$

$$E'_{FB} = E_{FB} - \omega_1 L_1\frac{\sin\alpha}{2} \quad \text{per unit width} \tag{h}$$

$$E_{FC} = E_{FA} \quad \text{and} \quad E'_{FC} = E_{FC} + \omega_1 L_1\frac{\sin\alpha}{2} \quad \text{per unit width} \tag{i}$$

$$E_{FD} = E_{FB} \quad \text{and} \quad E'_{FD} = E_{FD} + \omega_1 L_1\frac{\sin\alpha}{2} \quad \text{per unit width} \tag{j}$$

Case III

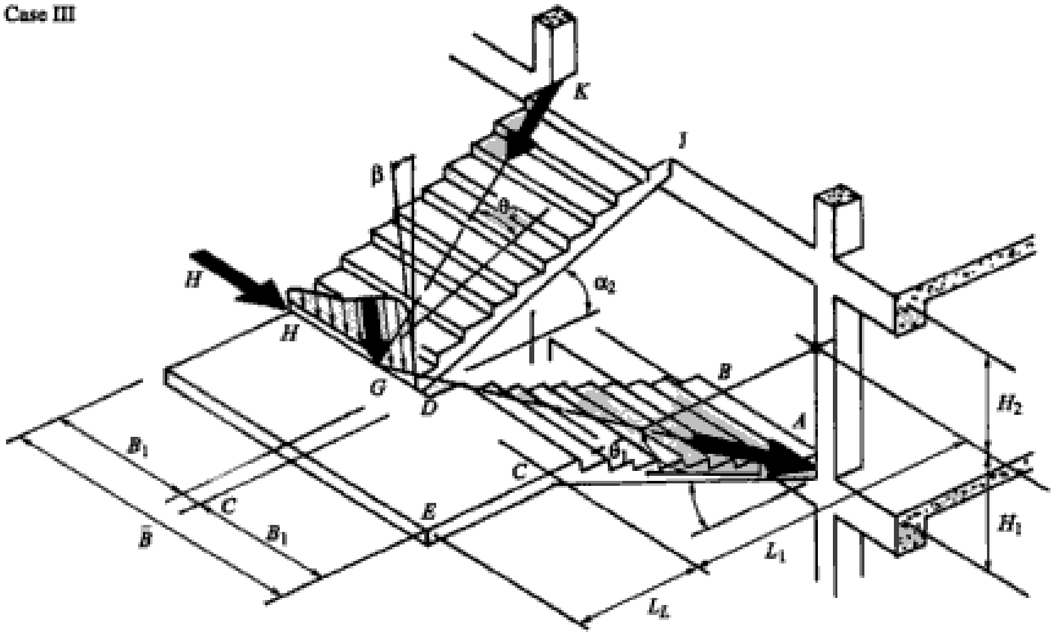


Figure 2.4 j(a). Floor slab restrained against horizontal movement.

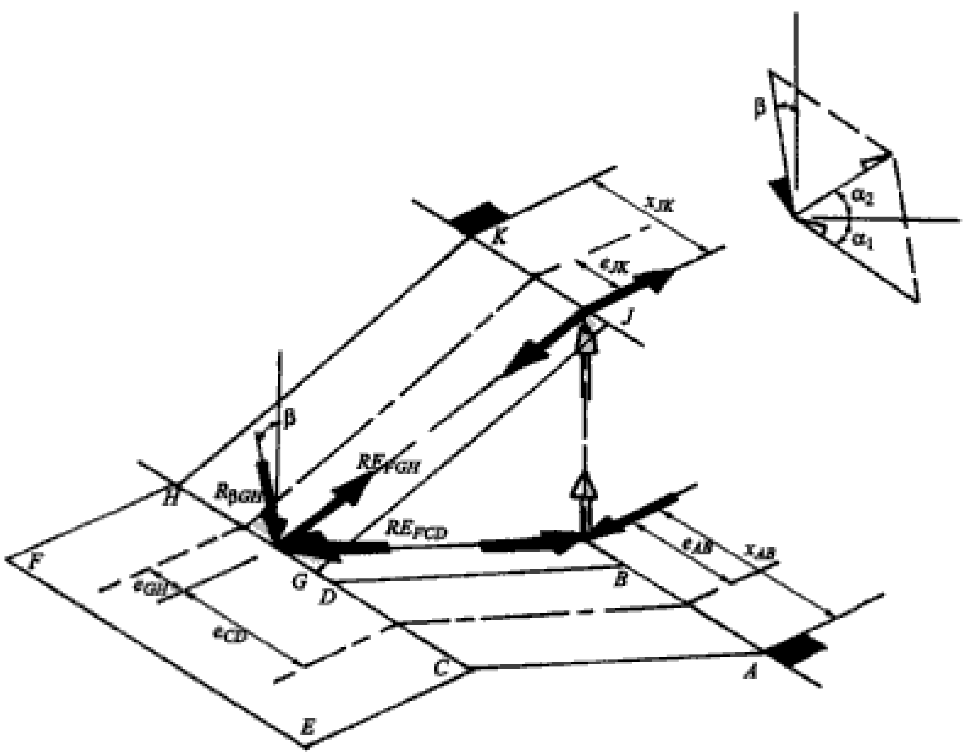


Figure 2.4 j(b). Boundary forces.

Case IV
M_1 acting at an angle β_1

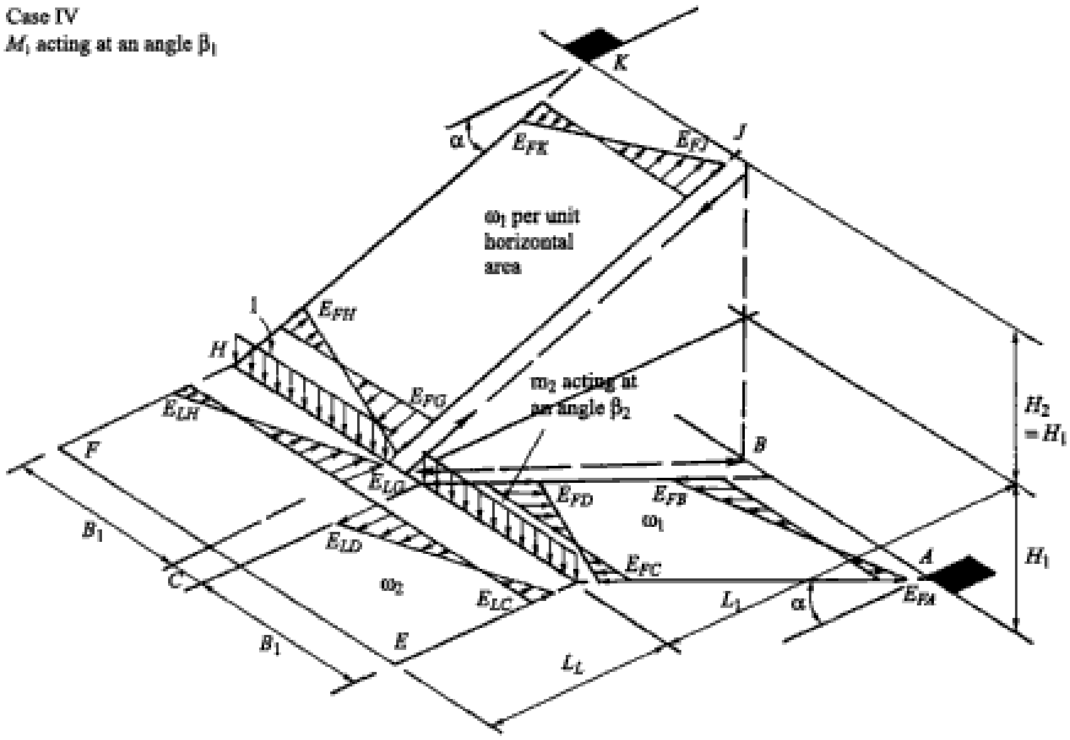


Figure 2.4 k(a). Membrane forces.

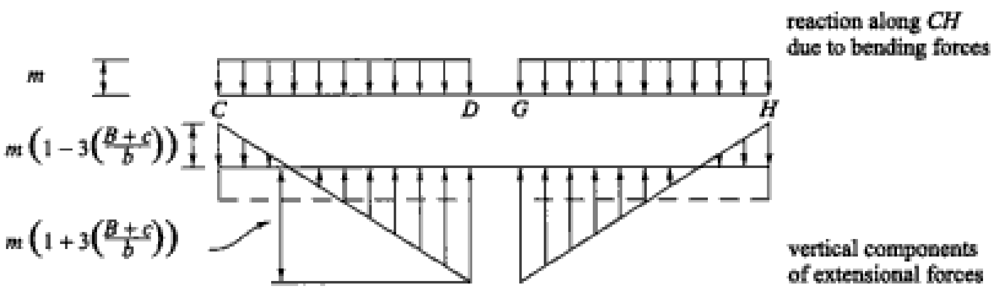


Figure 2.4 k(b). Forces acting in vertical plane.

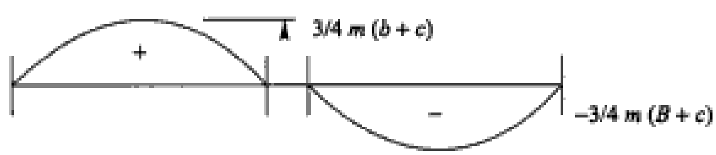


Figure 2.4 k(c). Shear forces acting in vertical plane.

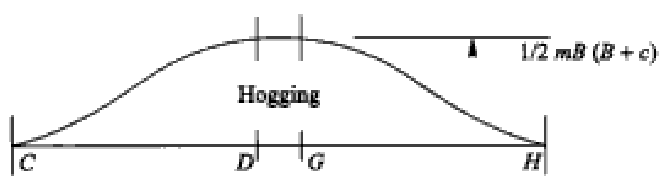


Figure 2.4 k(d). Bending forces acting in vertical plane.

Case V

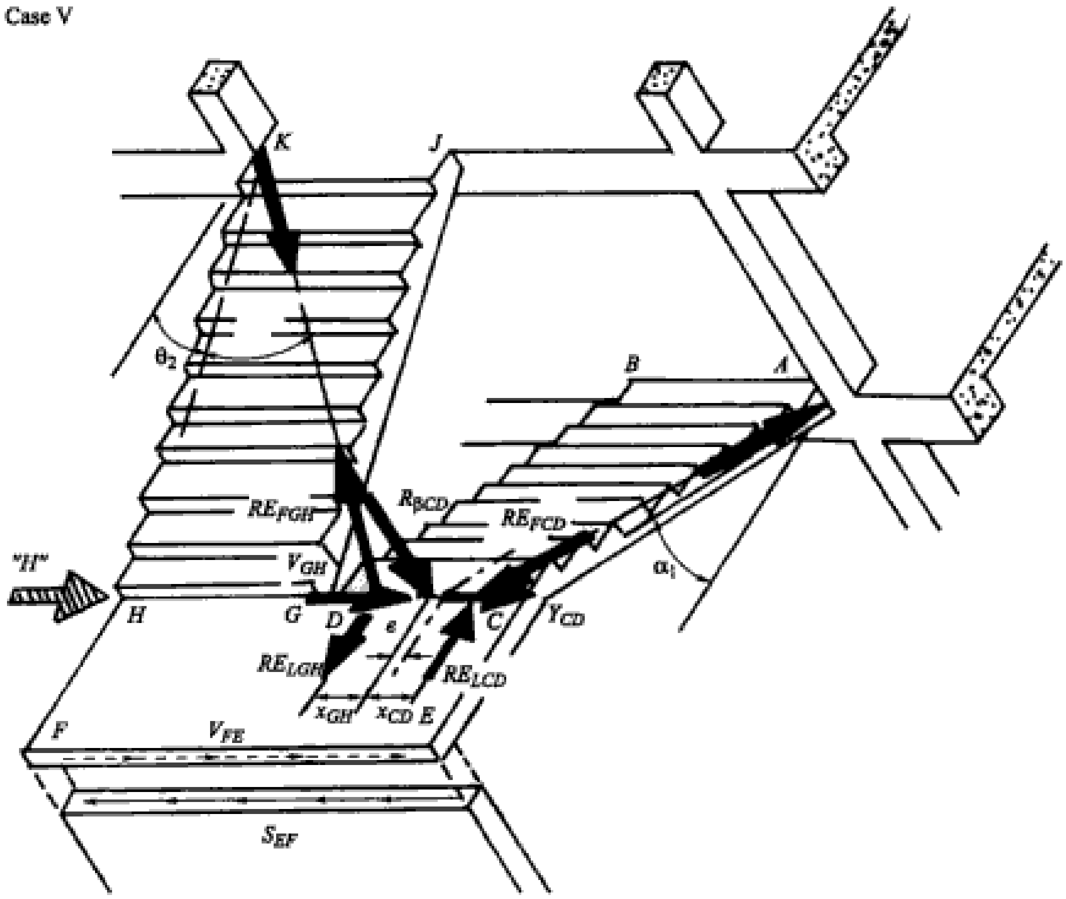


Figure 2.41(a). A staircase with support at mid landing.

$$E_{LC} = -\frac{m}{\tan\alpha}\left[1 - \frac{3(B+C)}{B}\right] \quad \text{per unit width} \tag{k}$$

$$E_{LD} = -\frac{m}{\tan\alpha}\left[1 + \frac{3(B+C)}{B}\right] \quad \text{per unit width} \tag{l}$$

$$E_{LG} = -E_{LD}, \quad E_{LH} = -E_{LC} \quad \text{per unit width} \tag{m}$$

$$E_{FG} = -E_{FD}, \quad E_{FH} = -E_{FC} \quad \text{per unit width} \tag{n}$$

$$E'_{FG} = E_{FG} - \omega_1 L_1 \frac{\sin\alpha}{2} \quad \text{per unit width} \tag{o}$$

$$E'_{FH} = E_{FH} - \omega_1 L_1 \frac{\sin\alpha}{2} \quad \text{per unit width} \tag{p}$$

$$E_{LJ} = -E_{FB}, \quad E'_{FK} = -E_{FA} \quad \text{per unit width} \tag{q}$$

$$E'_{FJ} = E_{FJ} + \omega_1 L_1 \frac{\sin\alpha}{2} \quad \text{per unit width} \tag{x}$$

$$E'_{FK} = E_{FK} + \omega_1 L_1 \frac{\sin\alpha}{2}$$

The unbalanced resultants of the primary extensional forces along CH are in this case in a vertical plane.

Table 2.2 (cont.).

Case V: a staircase with support at mid landing

Case V is similar to Case IV except the end of the intermediate landing is restrained horizontally and vertically as shown in Figure 2.4l(a).

Considering the equilibrium of the intermediate landing:

$$R_{BCD}\cos\beta = -RE_{FCD}\cdot\cos\Theta_1\sin\alpha_1 + RE_{FGH}\cdot\cos\Theta_2\sin\alpha_2 \qquad \text{(a)}$$
$$RE_{LGH} + R_{BCD}\sin\beta = -RE_{LCD}$$

where,

$$RE_{LGH} = RL_{FGH}\cos\Theta_2\sin\alpha_2$$
$$RE_{LCD} = RL_{FCD}\cos\Theta_1\sin\alpha_1$$
$$V_{FE} = V_{GH} + V_{CD} = RE_{FGH}\sin\Theta_2 - RE_{FCD}\sin\Theta_1$$
$$RE_{LGH} = X_{GH} - RE_{LCD}X_{CD} = V_{FE}\cdot a$$

$$RE_{FGH}\cos\Theta_2\sin\alpha_2\cdot X_{GH} + RE_{FCD}\cos\Theta_1\sin\alpha_1\cdot X_{CD}$$
$$\frac{3B}{2} + C - e - X_{GH} = \frac{L}{\cos\alpha_2}\tan\Theta_2 \qquad \text{(b)}$$
$$\frac{B}{2} + e - X_{CD} = \frac{L}{\cos\alpha_1}\tan\Theta_1$$

The internal extensional forces can consequently be determined.

2.6.2 *Siev's method (June 1962, October 1983)*

Plate analysis of multi-flight staircases

The space interaction of plates forms the basic theory of analysing staircases while assuming they are statically deteminate. The plates are divided into various horizontal shapes looking like trusses placed horizontally. The analysis is similar to that used in hipped plates. The line of the intersection between the flights and the landing is considered as a support and the load acting on it is resolved into forces in planes of the plates. Figures 2.5 and 2.8 show the various effects under symmetrical and antisymmetrical loadings. This method is very similar to Liebenberg's. Here torsional moments are then calculated as those moments causing compatibility in deformation.

Notation for the analysis

A	=	cross-sectional area of flight;
B, C, L_1, L	=	dimensions of stairs;
C	=	torsional rigidity;
D	=	overall depth of slab;
E	=	Young's modulus of elasticity;
f	=	stresses;
G	=	modulus of elasticity in shear;
I	=	moments of inertia;
I_b	=	moment of inertia of beam 3-10;
M	=	moment;
P	=	load at a point;

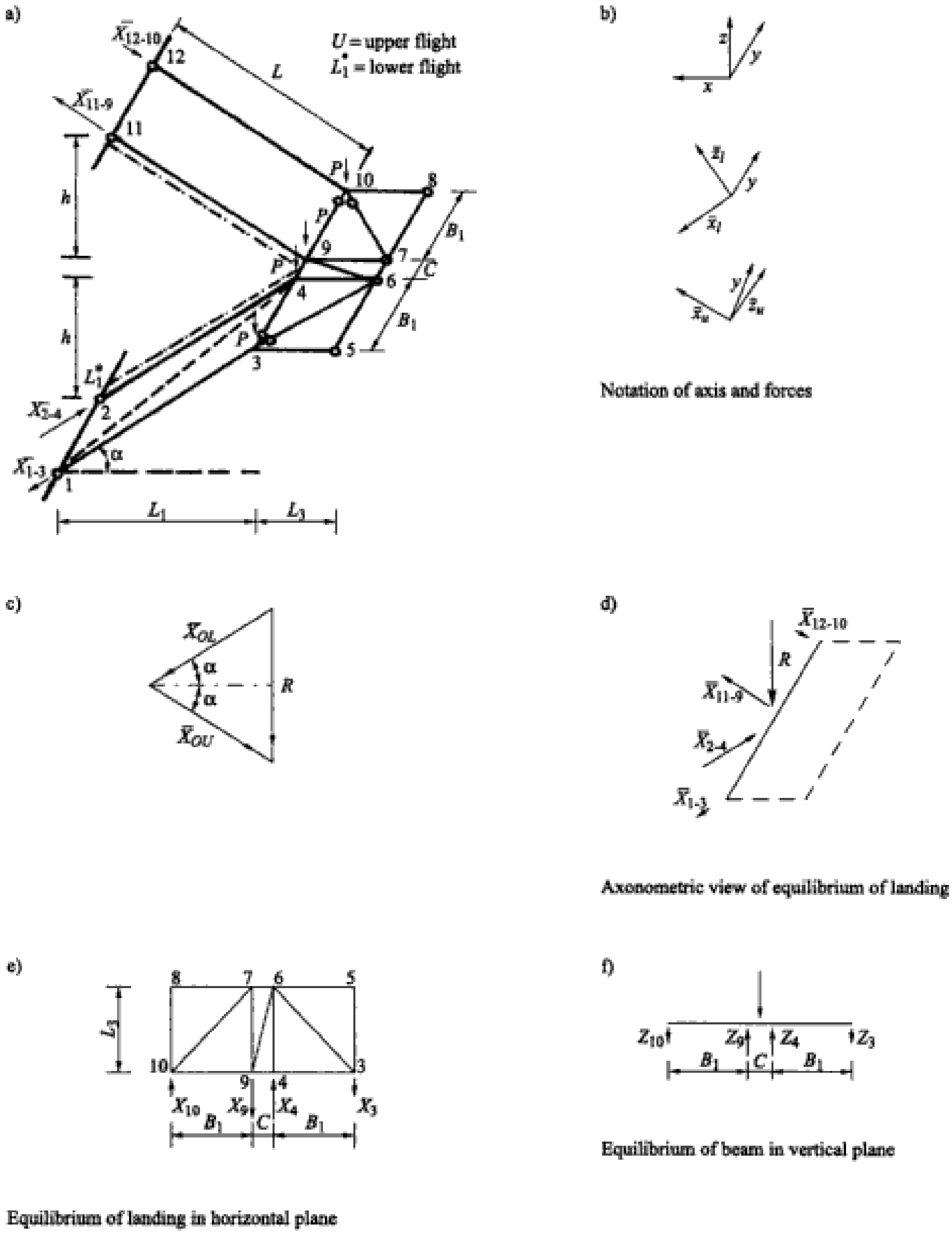


Figure 2.5. Equilibrium of space truss under symmetric load (Siev A. June 1962).

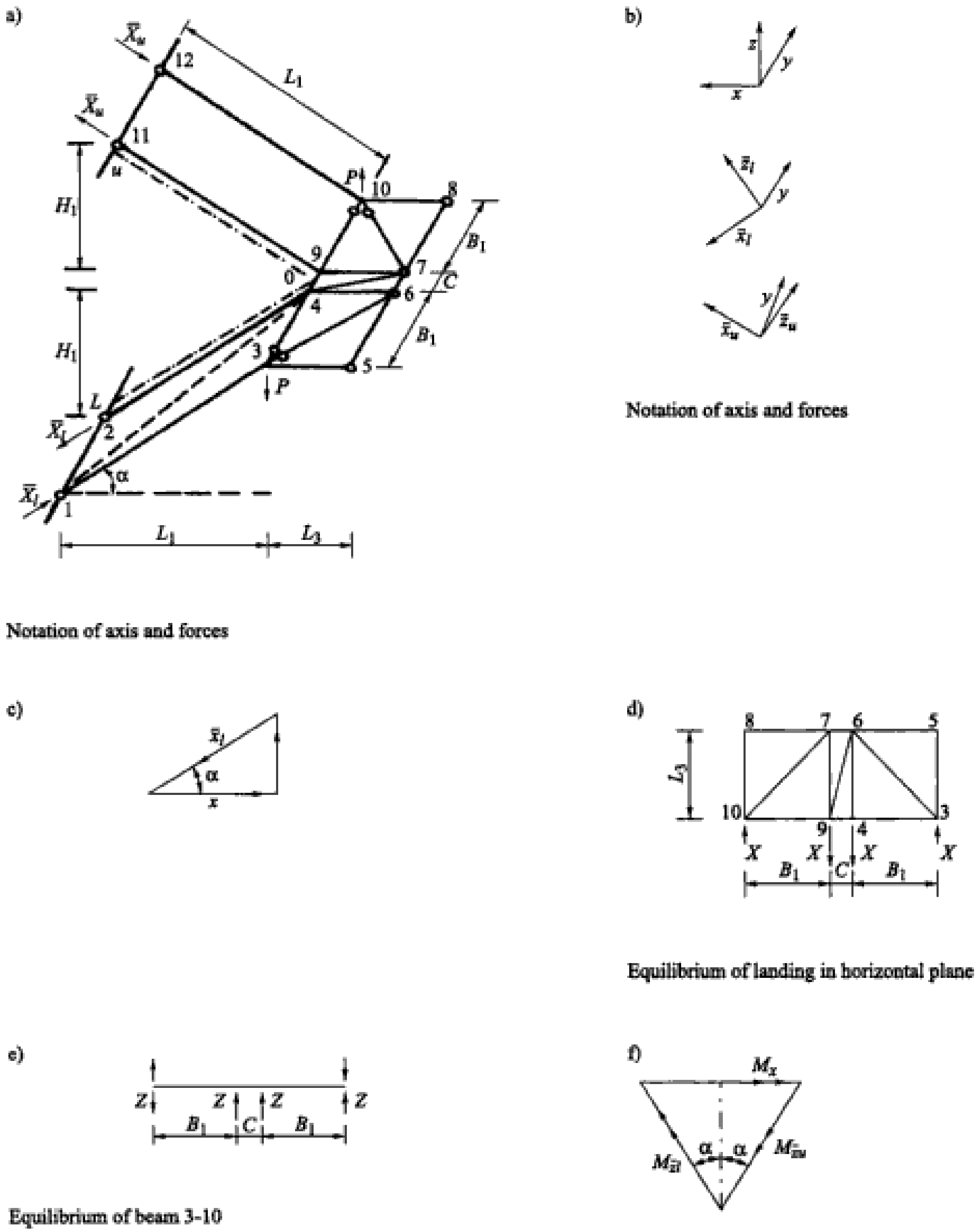


Figure 2.6. Equilibrium of space truss couple (Siev A. June 1962).

R	=	force (resultant);
R'	=	force resisted by primary stresses;
R''	=	force resisted by secondary stresses;
w	=	displacement normal to plate surface;
W	=	load W_L, W_D = live load and dead load, respectively;
X, Y, Z	=	forces in x, y, z directions, respectively;
$\overline{X}$, $\overline{Y}$, $\overline{Z}$	=	forces in $\overline{x}$, $\overline{y}$, $\overline{z}$ directions, respectively;

Greek

α	=	angle of slope of flight;
δ	=	deflection of a point, vertical;
ε	=	elongation of respective fibre; and
τ	=	torsional shear stresses at the flights.

Subscripts

u, l	=	upper flight, lower flight, respectively;
s, a	=	symmetric and antisymmetric, respectively;
x, y, z $\overline{x}$, $\overline{y}$, $\overline{z}$	=	indicate direction of moment, stress and so forth;
1, 2, 3 etc.	=	indicate point of moment, stress and so forth.

Basic analysis

Table 2.3 summarises the basic equations. The elements 1, 3, 5; 2, 4, 6; 7, 9, 11; 8, 10, 12 and 3, 4, 9, 10 each represent a single part with pin joints and are in vertical planes. Another element: a diagonal 1-4 dashed line is added to offer resistance to any horizontal forces. The unknown forces are represented by $\overline{x}$. The forces X and Z acting on the landing are the horizontal and vertical components of X such that

$X = \overline{x} \cos \alpha$ and $Z = \overline{x} \sin \alpha$

$$R'' \text{ (resistance to an additional load)} = R - R' \tag{2.63}$$

where R' is force (arbitrary) resisted by arbitrary load.

Moment in slab: moment

1. max. cantilever moment = 9.763 kN m
2. min. cantilever moment = 4.882 kN m
3. max. negative moment at floor levels (lines 1-2 and 11-12)
4. negative moments at floor levels for full load

$$= 12.24 + \frac{9.763}{2} = -7.3585 \text{ kN m}$$

5. max. positive moment = 5.02 kN m

Slab Reaction 3, 4 or 9, 10
$BR = 33.75$ kN

$$R = \frac{33.75}{1.22} = 27.664 \text{ kN}$$

Table 2.3. **Basic** analysis – Summary of equations.

Couple $M_X = P(2B + C)$ (a)

Example $\overline{X}_{1\text{-}3} = \dfrac{RC}{4B\sin\alpha}, \quad \overline{X}_{10\text{-}12} = -\dfrac{RC}{4B\sin\alpha} = -\overline{X}_{13}$ (b)

$$-\frac{R(C+2B)}{4B\sin\alpha} = +\overline{X}_{9\text{-}11} \tag{c}$$

$$\overline{X}_{OL} = \overline{X}_{OU} = \frac{R}{2\sin\alpha} \tag{d}$$

$$R = R' + R'' \quad \text{where, } R'' \ll R', \quad R \cong R' \tag{e}$$

$$\overline{X} = \frac{M_X}{2B\sin\alpha}, \quad X = \overline{X}\cos\alpha, \quad Z = \overline{X}\sin\alpha \tag{f}$$

$$F_{\overline{Z}9} = -F_{\overline{Z}4} = \frac{R'}{D\sin\alpha} = \frac{R'}{D\sin\alpha}\left(1 + 3\frac{B+C}{B}\right) \tag{g}$$

$$F_{\overline{Z}3} = \frac{R'}{D_f\sin\alpha}\left(1 - 3\frac{B+C}{B}\right) \tag{h}$$

M'_S = maximum mid-span primary negative bending moment for beam 3-10

$$= -BR'\frac{B+C}{2}$$

$$w'_1 - w'_4 = w'_{10} - w'_9 = \frac{6R'L(B+C)}{EBD\cos\alpha} \tag{i}$$

$$\delta''_3 = \delta''_4 = \frac{R'B^2(B+C)}{4EI_b}(C + 0.7B) - \frac{BM_X}{6EI_B}(3C + 2B) \tag{j}$$

$$w'''_3 = w'''_4 = \frac{M_X B1}{GC} = (w''_3 - w''_4) + (w^{iv}_3 - w^{iv}_4) + w'_3 - w''_3 - w^0_4 = \text{Eq. (i)}/\cos\alpha$$

$$R'' = \frac{D_f^2}{4L_1^2\sin^2\alpha}\left[1 + 3\left(\frac{B+C}{B}\right)^2\right]R' \tag{k}$$

$$M_{\overline{X}} = R'\left[\frac{\dfrac{(B+C)}{GLED_fB\cos\alpha} + \dfrac{B^2C(B+C)}{4EI_b\cos\alpha}}{\dfrac{BL}{GC} + \dfrac{12L\tan\alpha}{EB^2D_f} + \dfrac{BC}{2EI_b\cos^2\alpha}}\right] \tag{l}$$

$$w^{iv}_3 - w^{iv}_4 = w^{iv}_{10} - w^{iv}_9 = -12\frac{M_X H1}{ED_fB^2}\tan\alpha \tag{m}$$

under asymmetrical loading

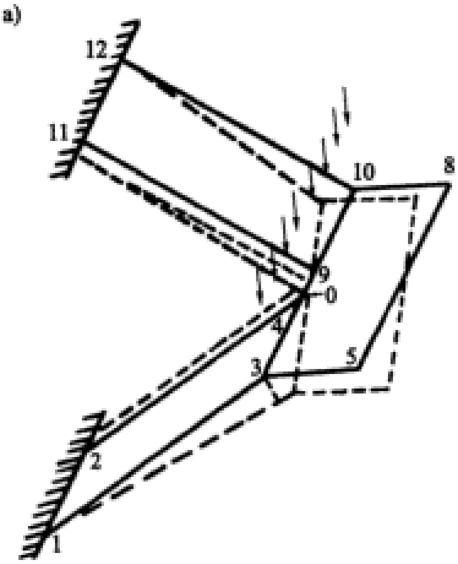
$$M'_X = B(B+C)R'_a \tag{n}$$

$$M_{\overline{Z}a} = \frac{M'_{Xa}}{2\sin\alpha} = \frac{B}{2}\frac{(B+C)}{\sin\alpha}R'_a \tag{o}$$

$$M''_{X,a} = \frac{GCB(B+C)}{EI_{\overline{Z}}(8\tan^2\alpha)}R'_a \tag{p}$$

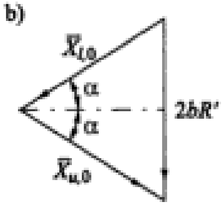
$$R''_a = \frac{GC}{EI_{\overline{Z}}}\frac{1}{4\tan^2\alpha}R'_a \tag{q}$$

a)

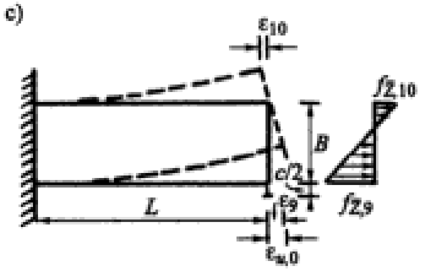


Stresses strains and deflections

b)

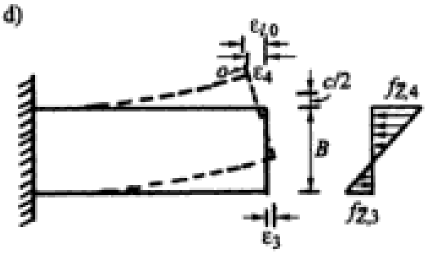


c)



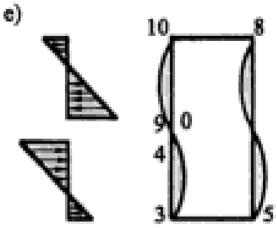
Strains in upper flight

d)



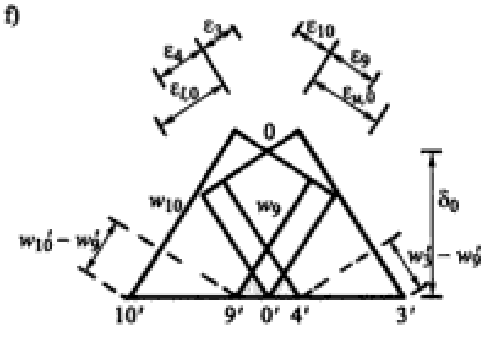
Strains in lower flight

e)



Deformation of landing

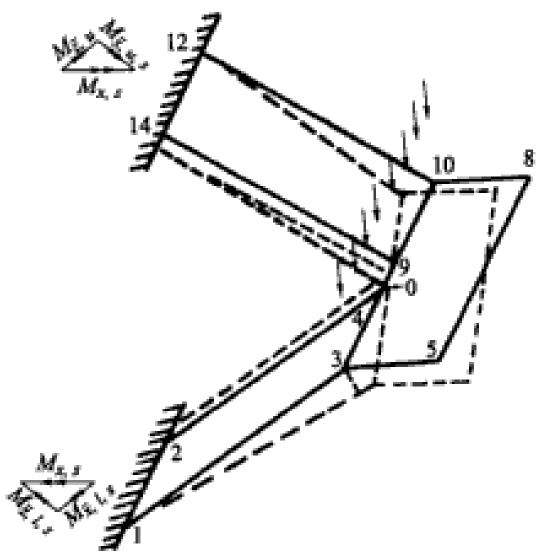
f)



Williot diagram of displacements due to strains in plates.

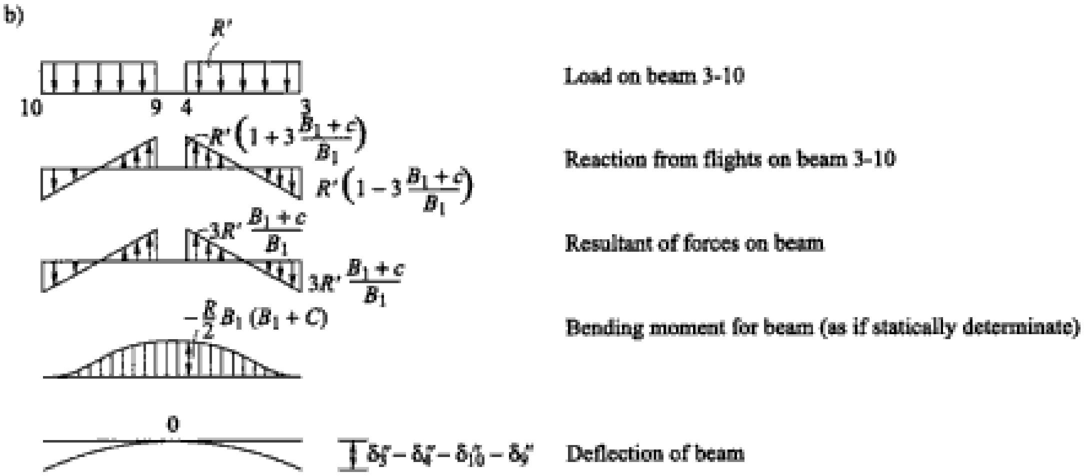
Figure 2.7. Stresses and displacements resulting from symmetrical loading due to strain in plates (Siev A. June 1962).

a)



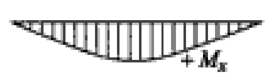
Displacement due to bending on beam 3-10

b)



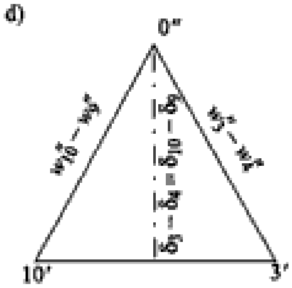
Loading and deflection of beam 3-10. Note: $B_1 + C = L_3$

c)



Positive moment distribution on beam due to torsional restraint

d)



Williot diagram

Figure 2.8. Displacement resulting from symmetrical loading due to bending of beam 3-10 (Siev A. June 1962).

EXAMPLE 2.3

Using the space interaction of plates for a multi-flight staircase and the following parameters, calculate moments, reactions and relevant stresses for a reinforced concrete staircase under both symmetrical and asymmetrical loads:

$L_1 = 2.9$ m, $H_1 = H_2 = 1.83$ m

$L = 3.414$ m

$B = 1.22$ m, $C = 0.3048$

$D_f = 114$ mm for flights

$D_f = 203$ mm for beam

$Wd =$ dead load on horizontal projection $= 4.8$ kN/m^2

$WL =$ imposed load $= 4.8$ kN/m^2

$G = 0.4$E

Assume the ends of the flights are completely fixed.

SOLUTION

A multi-flight staircase in concrete

$I_y = 151.7 \times 10^6$ mm^4, $I_z = 17{,}230 \times 10^6$ mm^4 Grade 30 concrete

$$C = \frac{BD_f^3}{3}\left(1 - \frac{0.63D_f}{3}\right) = 570 \times 10^6 \text{ mm}^4$$

$I_b = 426.0 \times 10^6$ mm^4

Beam 3-10 cross section 610 × 203

Equation (k) Table 2.3 for symmetrical loads gives

$R'' = 0.006363R'$ kN/m

The ratio between the secondary and the entire resistance:

$$\frac{R''}{R' + R''} = 0.0063$$

$M =$ secondary negative moments at the floor $= 0.0063R_s \times L_1 = 0.0063 \times 0.192 \times 2.9 = 0.035$ kN/m
$R_s = 0.192$ kN/m is acting on the primary system.

Table 2.3 Equations (g) and (h) is invoked replacing R' by R_s

$$f_{\bar{z}9} = -f_{\bar{z}9} = 2151 \text{ kN/m}^2$$

$$f_{\bar{z}3} = -f_{\bar{z}10} = 1241 \text{ kN/m}^2$$

With these stresses there will be an increase in reinforcement ration in the tension zone. In order that the torsional moment can be computed, $w_3 = w_4$, the relative displacements must be known.

$$E(w_3' - w_4') = 50.84 \text{ kN/m}$$

$$G(w_3'' - w_4''') = 0.486M_{x,s} \text{ kN m}$$

$$E(w_3^{iv} - w_4^{iv}) = -0.000641 \text{ kN/m}$$

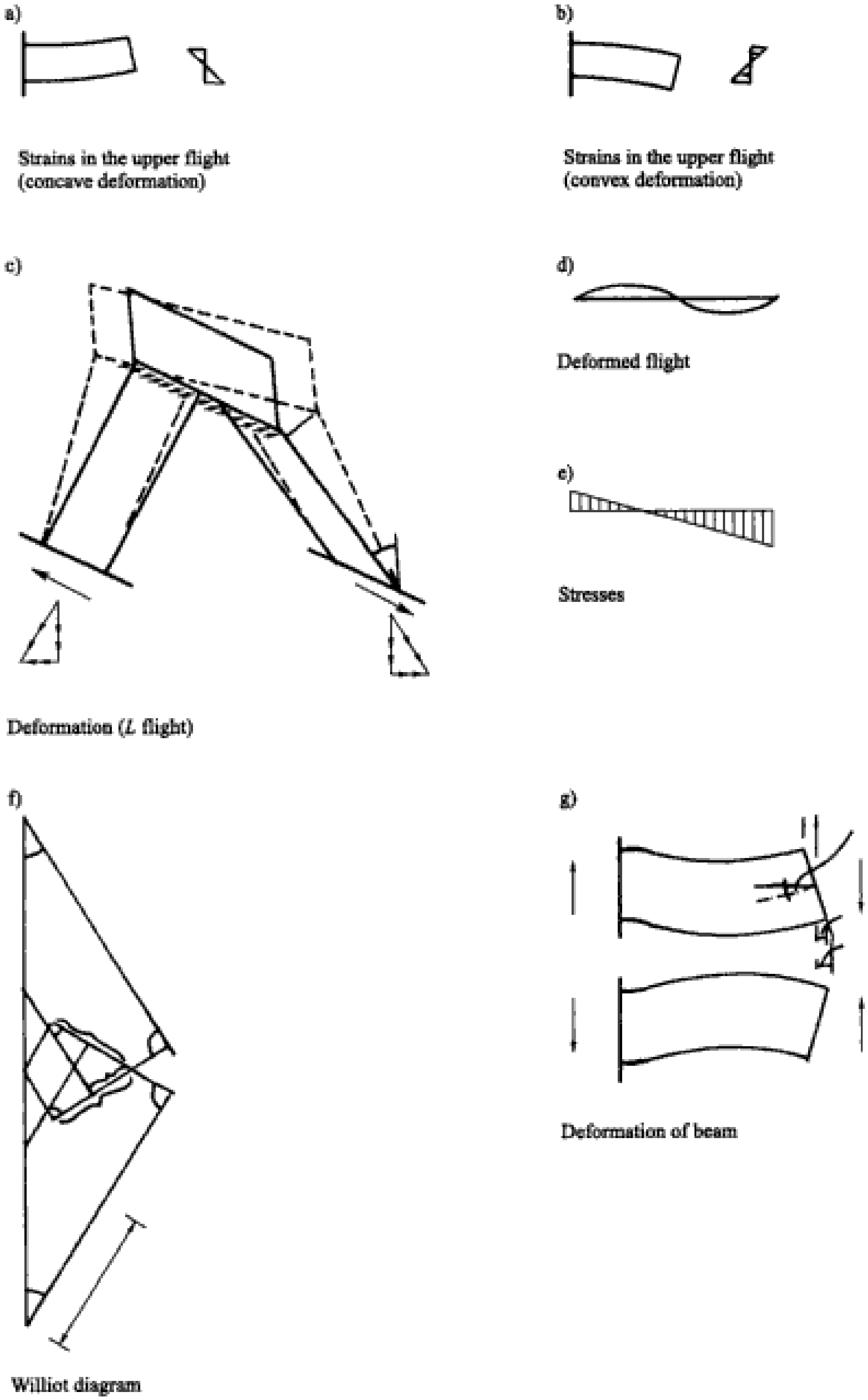


Figure 2.9. Stresses and displacements resulting from antisymmetrical loading.

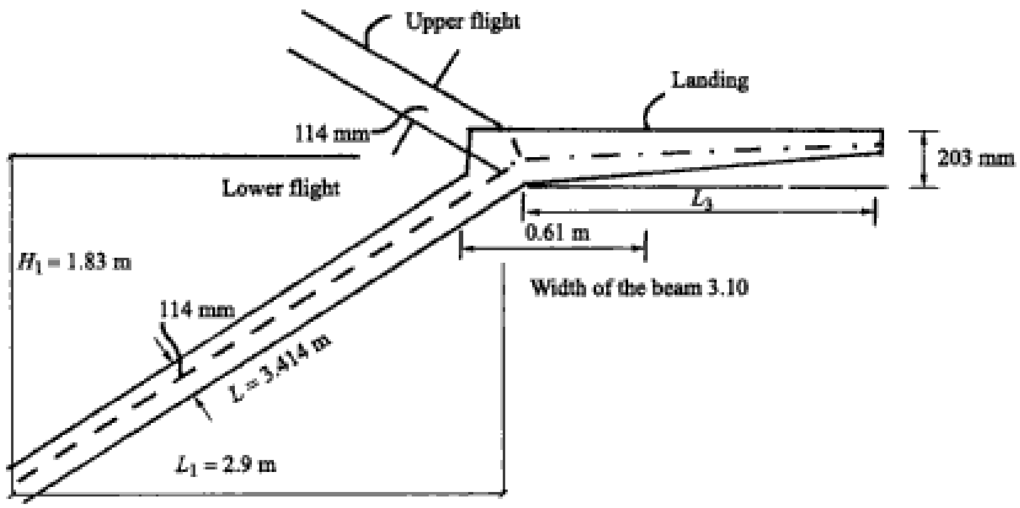


Figure 2.10. Typical cross-section through landing with variable depth.

substituting into Equation (j) Table 2.3

$M_x = 2.4521$ kN m, $\overline{m}_x = 2.096$ kN m

The deflection of the beam 3-10 contributes to the torque.

$$\text{Torsional shear stresses} = \tau = \frac{3M_{X1,S}}{BD_f^2} = 395\ \text{kN/m}^2 \text{ or}$$

$$= \frac{30m_{\overline{X},S}}{BD_f^2} = 395\ \text{kN/m}^2$$

If Grade 30 i.e. 30 MN/m^2 or 30×10^3 kN/m^2 concrete is used ($f_{cu} > f_c'$) stresses can be absorbed by the concrete easily. In this case no special torsional reinforcement is needed. This is the reason why torsional stresses are ignored in the Design Office Practice. This rigidity of the beam 3-10 is increased by using sufficient quantities of steel in the compressive zones, thus bringing about a reduction in the tensile stresses. Using Equation (g) Table 2.3

$M_{3,10} = 23.2$ kN m

The total primary moment = 25.65 kN m is adopted.

A symmetrical load
Using Equation (q) $R_a'' = 0.0052R_a'$

The full load Ra can therefore be assumed to produce primary stresses only (Eq. (e) Table 2.3).

Calculation of maximum stresses
(i) $w_d + 1/2w_L = 4.8 + 2.4 = 7.2$ kN/m^2 entire structure symmetrically loaded

(ii) one half of the structure with $+1/2w_L = 2.4$ kN/m^2 and the other half with $-1/2w_L = -2.4$ kN/m^2

Appropriate superposition gives the maximum stresses at each point. The previous values are multiplied by the load ratio $7.2/9.6 = 0.75$

$R_s = 0.145$ kN/m, $R_a \pm 0.05$ kN/m

$F_{\overline{Z}9,L} = 1614$ kN/m^2

$F_{\overline{Z}3,L} = 931.0$ kN/m^2

Using Equation (o) Table 2.3

$f_{\overline{Z}3,a} = 421$ kN/m$^2 = -f_{\overline{Z}9,a}$ under full load

$f_{\overline{Z}3\max} = 931.421 = -510$ kN/m^2

This value is 9% higher than stresses under full load on the entire staircase and is generally regarded as insignificant for practical purposes.

2.7 HELICAL STAIRS

2.7.1 *Introduction*

Recently, curved staircases have been constructed, supported only at the top and bottom. Although they are circular in plan projections, in elevation their description is helicoidal. Various analyses are available to solve such a complicated problem. From each analysis, torsional moments, bending moments, shear forces and axial thrusts are resulted. The geometry of each helical staircase affects the application of load and hence the results. This subject has been throughly reviewed in depth by various reseachers. In this text, only significant analyses are given which might assist researchers and practising engineers.

2.7.2 *Morgan's method (March 1960)*

Introduction to the method

This is one of the first methods produced for the helical stairs and is based on freely supported flights. A uniform load is assumed on a divided structure. Various moments including torsional moments are computed using a carefully cosidered geometry. The analysis also gives shears and axial thrusts.

Notation for the analysis

a_1, a_2	=	coefficients for redundant moment and force at midspan, respectively;
a_3	=	coefficient for vertical moment at fixed end;
B	=	width of the stair section;
E, G	=	moduli of elasticity of concrete in tension and compression and in shear, respectively;
H	=	horizontal redundant force at midspan;
D_f	=	total depth of stair section;

I_1, I_2	=	second moments of area of effective section of stair about horizontal axis $= 1/2BD_f^2$ and about axis normal to slope $= 1/2D_fB^2$, respectively;
J	=	polar second moment of area of effective section of stair $= K_2BD_f^3$ (for values of B greater than D_f);
K_2	=	torsional constant;
	=	$1/3 - 3.36D_f/16B[1-(D_f/B)^4/12]$;
$M_u = M_o$	=	redundant moment acting in a tangential plane at mid-span;
M_{nf}, M_{rf}	=	lateral moment (about axis normal to slope of stair) and vertical moment (about horizontal axis), respectively;
P_{nf}	=	thrust normal to tangent;
R_i, R_o	=	internal and external radi of the stair, respectively;
R_1	=	radius of centre-line of load;
R_2	=	radius of centre-line of steps $= 1/2(R_o + R_i)$;
V_{nf}, V_{rf}	=	shearing force across section of stairs and radial horizontal shearing force, respectively;
T_f	=	torsional moment;
w	=	total loading per unit projected length of centre-line of loads;

Greek

β	=	total area subtended by helix as seen in plan;
Θ	=	angle of subtended in plan measured from mid-point of stair;
ϕ	=	slope made by tangent to helix centre-line with respect to horizontal plane.

Basic analysis for a freely supported helical stair

This analysis is based on a freely supported flight. Figure 2.11 shows a typical helical staircase with various moments and reactions labelled on it. At the mid-point the angle Θ is positive when measured in a clockwise direction and is negative in an anti-clockwise direction. The strain energy concept again is applied. The loading is assumed to be symmetrical and hence the structure of the stair is divided at the centre.

The angle ϕ is constant.

$$\tan\phi = H_1/R_2\beta$$

where, H_1 = effective height of the stair; β = effective angle through which it turns; γ = cut off angle is taken to be 30 deg.

At any point O in the flight, Morgan developed the following equation: Vertical moment:

$$M_v = M_{ro} = wR_1R_2\frac{B\sin\alpha}{2} + wR_1^2(1+\cos\alpha) - HH_1\delta'\frac{\sin\Theta}{B} \quad (2.64)$$

Lateral moment:

$$M_{no} = \left[wR_1R_2(\cos\alpha - \Theta) - HH_1\delta'\frac{\cos\Theta}{B} - wR_1^2\sin\alpha\right]\sin\phi - HR_2\sin\Theta\cos\phi \quad (2.65)$$

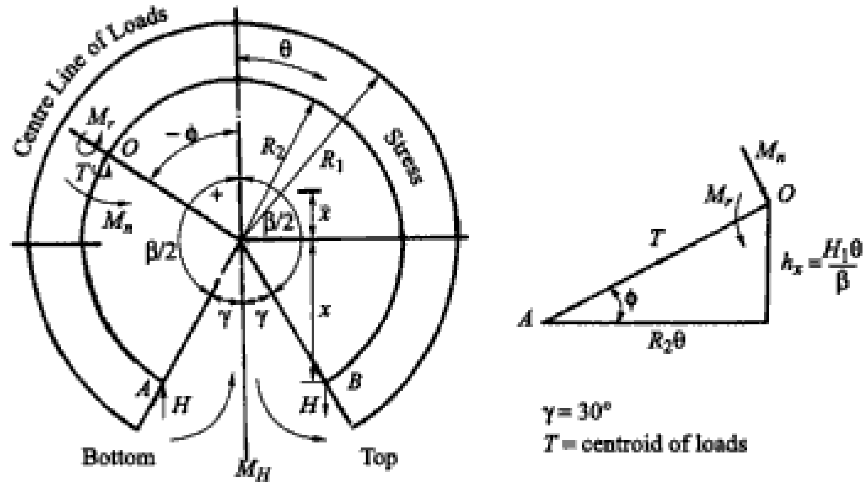


Figure 2.11. Helical stairs (Morgan's method).

Torsion:

$$T_o = \left[wR_1R_2(\cos\alpha - \Theta) - HH_1\delta' \frac{\cos\Theta}{B} - wR_1^2 \sin\alpha \right] \cos\phi + HR_2 \sin\Theta \sin\phi \tag{2.66}$$

Thrust normal to the tangent:

$$P_{no} = -H\sin\Theta\cos\phi - wR_1\Theta\sin\phi \tag{2.67}$$

Shearing force across the waist of the stairs:

$$V_{no} = wR_1\Theta\cos\phi - H\sin\Theta\sin\phi \tag{2.68}$$

Radial horizontal shearing force:

$$V_{HO} = H\cos\Theta \tag{2.69}$$

The vertical reactions at the simple support are:

$$wR_1B/2 \tag{2.70a}$$

$$\overline{x} = 2R_1\sin\beta/2B \tag{2.70b}$$

$$x = R_2\sin(\beta/2 - 90°) \tag{2.70c}$$

Then

$$HH_1 = w(\overline{x} + x) \quad \text{and} \quad M_{ho} = HR_2\sin\gamma \tag{2.71}$$

where,

$$\alpha = \Theta - \Delta, \quad \delta' = \Theta + \beta/2$$

EXAMPLE 2.4

Calculate various parameters for a freely supported helical stair using the following data:

$B = 300°$

R_i = internal radius = 0.195 m, width = 1.22 m

$R_2 = 1.524$ m, $H_1 = 3.2$ m

R_1 for the axis of the helix $= \dfrac{2.135^3 - 0.915^3}{2.135^2 - 0.915^2} \times \dfrac{2}{3} = 1.603$ m

SOLUTION *F*

Reely supported helical stair

reactions at the end of flight = 61.38 kN

$H = 64.45$ kN, M_{HO} at $O = 47.6$ kN m

Substituting these values in Equations (2.64) to (2.69) various results are tabulated in Table 2.4.

Basic analysis of a stair flight with fixed ends

Again Morgan (March 1960) adopted the Strain Energy concept in deriving various expressions for a helical staircase flight with the far ends fixed. The origin is kept at C as shown in Figure 2.12. Bending moments to the right of this point are assumed to be positive when acting in an anti-clockwise direction when viewed along their axes towards point O. The reverse of this is treated as negative. If M'_v is the bending moment acting in the tangential plane at C, the values of M_{rf} and M_{nf} and T_f are computed as:

$$M_{rf} = M'_v \cos\Theta + HR_2\Theta\sin\Theta\tan\phi - wR_1^2(1-\cos\Theta) \tag{2.72}$$

$$M_{nf} = (M'_v \sin\Theta - HR_2\Theta\cos\Theta\tan\phi - wR_1^2\sin\Theta - wR_1R_2\Theta)\sin\phi - HR_2\sin\Theta\cos\phi \tag{2.73}$$

$$T_f = (M'_v \sin\Theta - HR_2\Theta\cos\Theta\tan\phi + wR_1^2\sin\Theta - wR_1R_2\Theta)\cos\phi + HR_2\sin\Theta\sin\phi \tag{2.74}$$

$$\frac{\partial U}{\partial M} = 0$$

$$= \int_0^{B/2} \left(\frac{\Delta M_{rf}}{\Delta M} \cdot \frac{M_{rf}}{EI_1} + \frac{\delta M_{nf}}{\delta M} \cdot \frac{M_{nf}}{EI_2} + \frac{\delta T_f}{\delta M} \cdot \frac{T_f}{CJ} \right) R_2\, d\Theta \tag{2.75}$$

$$\frac{\partial U}{\partial H} = 0$$

Table 2.4. Summary of results.

Parameters	Degrees (Θ)										
	−15°	−120°	−90°	−60°	−30°	0°	30°	60°	90°	120°	150°
M_{ro} (kN m)	0	−80	−72.30	−11.00	35.60	77.85	50.0	−11.00	−72.30	−80.00	35.60
M_{no} (kN m)	155.68	280	347	311	187	0	−187.00	−311	−347	−280	−155.68
T_o (kN m)	−55.60	−33.36	11.00	33.36	31.00	0	−27.00	−38.00	−11.00	33.36	55.60
P_{no} (kN)	53.40	66.72	72.30	51.20	33.36	0	−33.36	−51.20	−72.30	−66.72	−53.40
V_{no} (kN)	−44.50	−29.00	−11.00	−6.67	−0.23	0	0.23	6.67	11.00	29.00	44.50
V_{ho} (kN)	−57.80	−33.36	0	33.36	55.60	62.30	33.36	55.60	0	−33.36	−57.80

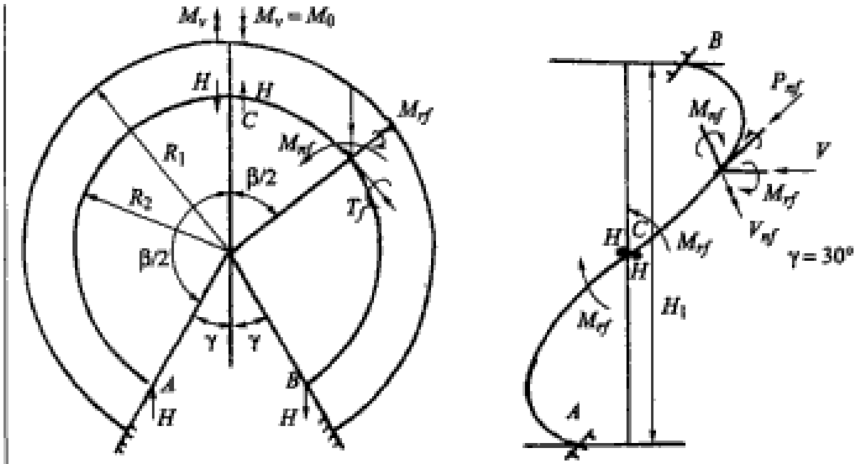


Figure 2.12. A helical staircase with fixed ends.

$$= \int_0^{\beta/2} \left(\frac{\delta M_{rf}}{\delta H} \cdot \frac{M_{rf}}{EI_1} + \frac{\delta M_{nf}}{\delta H} \cdot \frac{M_{nf}}{EI_2} + \frac{\delta T_f}{\delta H} \cdot \frac{T_f}{CJ} \right) R_2 \, d\Theta \tag{2.76}$$

The other values are:

$$P_{nf} = \text{thrust} = -H \sin\Theta \cos\phi - wR_1\Theta \sin\phi \tag{2.77}$$

V_{nf} = shear force across the waist of the stairs

$$= wR_1\Theta\cos\phi - H\sin\Theta\sin\phi \tag{2.78}$$

$$V_{Hf} = \text{radial horizontal shearing force} = H\cos\Theta \tag{2.79}$$

From Equations (2.75) and (2.76)

$$\begin{aligned}
&\frac{GJ}{EI_1}\left[m + \frac{1}{2}\sin 2\Theta (M_v' + wR_1^2) - wR_1^2 \sin\Theta - \overline{K} H R_2 \tan\phi\right] \\
&\quad + S\left[m(M_v' + wR_1^2) + nwR_1R_2 + \overline{K} H R_2 \tan\phi\right] \\
&\quad + HR_2 \sin\phi\cos\phi m\left(1 - \frac{GJ}{EI_1}\right) = 0
\end{aligned} \tag{2.80}$$

$$\begin{aligned}
&\frac{GJ}{EI_1}\left[nwR_1^2 - \overline{K}(M_v' + wR_1^2) + \frac{HR_2}{2}\tan\phi\left(\frac{\Theta^3}{3} - \frac{\Theta^2 \sin 2\Theta}{4} - \overline{K}\right)\right] \\
&\quad + S\left[\overline{K}(M_v' + wR_1^2)\right] + wR_1R_2(\Theta^2\sin\Theta + 2n) \\
&\quad + \frac{HR_2}{2}\tan\phi\left(\frac{\Theta^3}{3} + \frac{\Theta^2\sin 2\Theta}{4} + \overline{K}\right) + \left(S - \frac{GJ}{EI_2}\right) \\
&\quad + x\left[m(M_v' + wR_1^2) + nwR_1R_2 + \overline{K} H R_2 \tan\phi\right] \\
&\quad + \overline{K} H R_2 \sin\phi\cos\phi\left(1 - \frac{GJ}{EI_2}\right) \\
&\quad + mHR_2\cos^2\phi\left(\tan\phi + \frac{GJ^2\cot\phi}{EI_2}\right) = 0
\end{aligned} \tag{2.81}$$

where,

$$\begin{aligned}
&\overline{K} = \frac{\Theta\cos^2\Theta}{4} - \frac{\sin 2\Theta}{8}, \quad m = \frac{\Theta}{2} - \frac{\sin 2\Theta}{4}, \\
&n = \Theta\cos\Theta - \sin\Theta, \quad S = \cos^2\Theta + \frac{GJ}{EI_2}\sin^2\Theta
\end{aligned} \tag{2.82}$$

EXAMPLE 2.5

Determine the values of M'_v, M_{rf}, M_{nf}, T_f, V_{hf}, H and P_{nf} for Θ varying from 0 to $\pm$120 deg, using the following data:

ϕ	=	the angle of inclination of the helical stairs to the horizontal = 25 deg;
β	=	240 deg;
B	=	1.22 m;
D	=	thickness including tread and riser = 152 mm;
R_1	=	1.603;
R_2	=	1.524 m;
R_i	=	radius of the inside of the stairs = 0.915 m;
$H_1 = H_t$	=	effective height 3.2 m;
w	=	uniform distributed load = imposed load + dead load = 2.873 kN/m^2+ 3.59 kN/m^2 = 6.463 kN/m^2;
G/E	=	0.429

SOLUTION

Analysis of a helical stairs with far ends fixed

$R_1/R_2 = 1.05$;
$B/D = 12.2$ introducing data and these values
$M'_v = -2.0$ kN m;
$H = 191$ kN

The above equations are solved (Θ varying 0 to $\pm 120°$) and the following table summarises the results.

The overall impression from these examples is that boundary conditions play an important role in the assessment of various moments and reactions.

Limiting criteria for the design ultimate torsional moment

Elastic theory, using gross concrete area of structures, often leads to uneconomical design. Stairs in particular, when subjected to significant torsional moment, become uneconomical unless the design ultimate torsional moment is limited to a maximum value. It should be equal to $0.33\, f_c\, x^2 y/3$ N mm $= T_o$, where x and y are, respectively, the shorter and longer dimensions and f_c is the cylindrical compressive strength of concrete. Generally $f_c = 0.87 f_{cu}$ = cubic strength of concrete. For a fixed ended stair flights, while keeping the analysis given in Section 2.5 the value of T_o is constant. Substituting T_f from Equation (2.74) and expressing H in terms of M'_v and $(T_f)_\Theta = \gamma_1 = T_o$ the horizontal thrust H can be written as

$$H = \lambda_1 + \lambda_2 M'_v \tag{2.83}$$

Table 2.5. Summary of results.

Parameters	Degrees (Θ)								
	−120°	−90°	60°	30°	0°	30°	60°	90°	120°
M_{rf} (kN m)	−6.50	0	1.50	−1.5	−2.30	0	1.50	−1.50	−6.50
M_{nf} (kN m)	27.12	30.20	27.12	14.40	0	−16.30	−27.12	−30.12	−27.12
T_f (kN m)	−2.10	−2.20	−0.20	0.30	0	−0.30	−0.20	0	0
P_{nf} (kN)	26	25.30	19.30	11.00	0	−45	−19.30	−25.30	−25.60
V_{nf} (kN)	−17.00	−9.30	−4.50	−2.25	0	2.25	4.50	−9.30	17.00
V_{Hf} (kN)	−8.75	8.00	8.90	12.50	17.80	12.5	8.90	−2.25	−8.90

Note: These results can now be compared with Example 2.5 for the same input data up to 128. The results show that a freely supported stair can produce results different from the fixed ended one. Where supports are semi-rigid, the average results of the two are acceptable.

where,

a) $$\lambda_1 = \frac{T_o - \cos\phi(wR_1^2 \sin\gamma_1 - wR_1R_2\gamma_1)}{R_2(1+\gamma_1\cot\gamma_1)\sin\phi_1\sin\gamma_1} \tag{2.84}$$

b) $$\lambda_2 = \frac{\cot\phi}{R_2(1+\gamma_1\cot\gamma_1)}$$

where, γ_1 = a particular value of Θ for T_f maximum; γ_2 = a particular value of Θ for $M_{rf} = 0$ point of inflection; w = design ultimate load per unit projected length of the centre line of the load.

Table 2.5 gives other parameters given by Rangan et al. (1978) and Rajagopalan (1973).

EXAMPLE 2.6

Assuming the following data, determine M_v' and H and other parameters for the data given in Example 2.5

$R_1 = 1.603$ m, $R_2 = 1.524$ m, $R_1/R_2 = 1.05$,

$f_c' = 20$ MN/m^2 or Mpa, $T_o = 6.31$ kN m

$\gamma_1 = 2.36$ radians $= 0.38B$, $\phi = 25°$, $w = 6.463$ kN/m^2

SOLUTION

Torsional limitation for design ultimate load

$$\lambda_1 = \frac{6.31 - 0.90631(6.463 \times 1.1236 \times 0.68 - 6.463 \times 1.603 \times 1.524 \times 2.39)}{1.524(1 + 2.39 \times 1.07)0.4226 \times 0.68}$$

$$= 23.13$$

$$\lambda_2 = \frac{2.1445}{1.524(1 + 2.39 \times 1.07)} = 0.3956$$

Equations for helical stairs based on Rajagopalan method

At $\Theta = \gamma_2$ the moment M_{rf} is assumed zero. Substituting for H_o at 0 from Equation (2.83), Equation (2.72) gives

$$M_v' = \frac{[wR_1^2(1-\cos\gamma_2) - \lambda_1 R_2\gamma_2\tan\phi\sin\gamma_2]}{\cos\gamma_2 - \lambda_2R_2\gamma_2\tan\phi\sin\gamma_2} \tag{a}$$

To find γ_1 and γT_f is maximum by γ_1 and substituting into Equation (2.74), the following expression is developed.

$$M_v'\cos\gamma_1 + H_oR_2\tan\phi\gamma_1 + wR_1R_2 = 0 \tag{b}$$

when $M_{rf} = 0$; when $\Theta = \gamma_2$, Equation (2.72) gives

$$0 - M_v'\cos\gamma_2 + H_oR_2\gamma_2\tan\phi\sin\gamma_2 - wR_1^2(1-\cos\gamma_2)$$

or

$$w = \frac{M_v'\cos\gamma_2 + H_oR_2\gamma_2\tan\phi\sin\gamma_2}{R_1^2(1-\cos\gamma_2)} \tag{c}$$

For optimum value of γ_2; $\gamma\omega/\gamma_1\gamma_2 = 0$ the following expression is developed

$$M_v' - H_oR_2\tan\phi[1 + \gamma_2\cot\gamma_2(1-\cos\gamma_1) + \gamma_2\sin\gamma_2] = 0 \tag{d}$$

substituting for H_0 from Equation (2.83), Equation (d) can be written as

$$M_v' = \frac{\lambda_1}{\lambda_2 + \bar{x}} \tag{e}$$

where

$$\bar{x} = \frac{1}{R_2} \tan\phi\left[1 + \gamma_2 \cot\gamma_2(1 - \cos\gamma_2) + \gamma_2 \sin\gamma_2\right] \tag{f}$$

For values of $\lambda_1 = 23.13$ and $\lambda_2 = 0.3956$ Equations (a) (Table 2.5) and (e) give

$$M'_v = \frac{[6.463(1.603)^2(1 - \cos\gamma_2 - 23.13 \times 1.524\gamma_2 \times 0.46631 \sin\gamma_2)]}{\cos\gamma_2 - 0.3956 \times 1.524\gamma_2 \times 0.46631 \sin\gamma_2}$$
$$= 0$$

or

$$M'_v = \frac{[26.6074(1 - \cos\gamma_2 - 16.4375\gamma_2 \sin\gamma_2)]}{\cos\gamma_2 - 0.2811\gamma_2 \sin\gamma_2}$$

Again from Equation (e)

$$M'_v = \frac{\lambda_1}{\lambda_2 + \bar{x}}$$
$$= \frac{23.13}{0.3956 + 0.3062[(1 + \gamma_2 \cot\gamma_2)(1 - \cos\gamma_2) + \gamma_2 \sin\gamma_2]}$$

The interaction between these two values of M'_v from Equations (a) and (e) gives

$$\gamma_2 = 2.313 \quad \text{and} \quad M'_v = -7.841 \text{ kN m}$$

Hence $H_0 = 23.13(-7.841 \times 0.3956) = 26.23$ kN.

Equations (2.72) to (2.79) are invoked for parameters such as M_{rf}, M_{nf}, T_f, P_{nf}, V_{nf} and V_{Hf}. Similar calculations are made as given in Example 2.5.

2.7.3 *Cohen's method (May 1955)*

Introduction

Here a comprehensive package is given for the analysis of determinate and indeterminate conditions of helical stairces. The distributed load $w(s)$ can be non-uniform, with a non-uniform bending moment $M(s)$ per unit length of the curve. General equations of equilibrium are related to three loaded axes. An element of an arc is considered for a twisted curve. For a statically indeterminate staircase, the equations of equilibrium are not sufficient. In addition, equations of deformation and angular rotation at any point needed to be considered. The determinate beam staircase involves cantilevers with supported beams and beams with three supports. The indeterminate considers cases where both ends are fixed or one end is fixed and the other is pinned.

Notation for the analysis

a	=	radius;
c	=	constant;
D	=	sections;
H_1	=	height;
K	=	$\sin\phi/a$;
L, m, n etc.	=	direction cosine;
T, N, B	=	principal lines;
$\bar{t}$, $\bar{n}$, $\bar{b}$	=	normals in various directions;
M	=	moments;
M_t	=	twisting moment;

P	=	forces;
S	=	length;
V	=	projections;
$W_{x,y,z}$	=	load unit/length;
W	=	load;

Greek

α, Θ, Θ', ψ, ϕ = parameters defined.

General differential equations of equilibrium (determinate)
Table 2.6 shows a twisted curve related to the system of coordinates Ox, y, z

$$x = x(s), \quad y = y(s), \quad z = z(s) \tag{2.85}$$

Table 2.7 gives a summary of equations of equilibrium for a freely supported helical staircase. These are then adopted by including the effects of uniform and non-uniform loadings and distributions of their respective moments. The final expression are derived for moments, and other reactions.

EXAMPLE 2.7
Analyse a freely supported helical staircase, with timber treads fixed to a reinforced concrete helical beam. Use the following data:

$a = 0.85$, $\quad a_1 = 0.525$ m, $\quad a_2 = 1.56$ m, $\quad \Theta' = 240°$, $\quad H_1 = 3.375$ m

timber treads: 1.05 m long and 0.05 m thick
R. C. beam = 0.338 m wide and 0.213 m deep

SOLUTION
A helical staircase with timber treads and R. C. helical beams.

Evaluation of parameters:

$C = H_1/\Theta' = (3 \times 3.75)/4\pi = 0.8054$, $\quad \cot\phi = c/a = 0.92$,
$\phi = 47°20'$
$\sin\phi = 0.735$, $\quad \cos\phi = 0.678$, $\quad \tan\phi = 1.085$
$K = \sin\phi/a = 0.84$
$S =$ helix length $= \Theta'/K \approx 5$ m

$$R = \text{centroid of each thread} = \frac{2}{3}\frac{(a_2^3 - a_1^3)}{a_2^2 - a_1^2} = 1.128 \text{ m}$$

Weights:

weight of the beam = 8.52 kN, $\quad w$ total = 26.027 kN
weight of the treads = 17.507 kN
w/metre length = 26.027/5.0 = 5.205 kN/m
$m_1 = 17.507/5.00(1.128 - 0.875) = 0.886$
$C_1 = -8.7789456$, $\quad C_2 = -5.1358775$ $\quad C_3 = -0.187853$
$C_4 = -8.3487387$, $\quad C_5 = -0.6259467$, $\quad C_6 = -8.2269908$

from 0 to angles all values for T_t, T_n, T_b, M_t, M_n, and M_b are calculated. The values from $2\pi a/3$ to $4\pi a/3$ will have the same values as those given by Θ' but with opposite signs. These values are summarised for various angles of Θ'.

Table 2.6. Final results.

	0	$\pi a/6$	$\pi a/3$	$\pi a/2$	$2\pi a/3$	$5\pi a/2$	πa	$7/6(\pi a)$	$4\pi a/3$
T_f (kN)	−13.10	−11.544	−8.674	−4.670	0	4.67	8.674	11.544	13.10
T_n (kN)	?	0	3.358	5.810	6.716	5.810	3.358	0	−3.358
T_b (kN)	−5.631	−2.624	−0.845	−0.116	0	0.116	0.845	2.624	5.631
M_t (kN m)	11.50	8.520	4.604	1.713	0	−1.713	−4.604	−8.520	−11.50
M_n (kN m)	0	−6.623	*6.94 max −5.613	−20.06	0	−2.006	−5.613	−6.623	0
M_b (kN m)	12.50	18.81	at 0.218 +18.882	11.75	0	−11.750	−18.882	18.881	−12.50

Note: *19.771 max at $\pi a/4$.
The maximum or minimum values by differentiation with respect to Θ'. The points of intersection are found from the second differentials of Equations (1).

Table 2.7. Summary of equations for determinate helical stairs.

$$\rho = \text{radius of curvature} = \frac{dS}{d_1\phi_1} \tag{a}$$

$$\tau_r = \text{radius of torsion} = \frac{dS}{d\phi_2} \tag{b}$$

Direction cosine:
Three principal lines:

$$\begin{aligned} \text{For}\quad & T\colon \alpha_1,\ \beta_1 \text{ and } \gamma_1 \\ & N\colon l_{1s},\ m_1 \text{ and } n_1 \\ & B\colon \lambda_1,\ \mu_1 \text{ and } \nu_1 \end{aligned} \tag{c}$$

Ignoring small angles:

$$\cos d\phi_1 = \cos d\phi_2 = 1 \qquad \sin d\phi_1 = \phi_1;\ \sin d\phi_2 = \phi_2$$

$$\frac{\cos d\phi_1}{2} = \frac{\cos \phi_2}{2} = 1 \qquad \frac{\sin d\phi_1}{2} = \frac{d\phi_1}{2};\quad \frac{\sin d\phi_2}{2} = \frac{d\phi_2}{2}$$

Now the following geometry is established:

$$\alpha_1 = \frac{dx}{ds},\quad \beta_1 = \frac{dy}{ds},\quad \gamma_1 = \frac{dz}{ds}$$

$$l_1 = \rho\frac{d^2x}{ds^2},\quad m_1 = \rho\frac{d^2y}{ds^2},\quad n_1 = \rho\frac{d^2z}{ds^2}$$

$$\lambda_1 = \rho\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right),\quad \mu_1 = \rho\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right) \tag{c}$$

$$V_1 = \rho\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)$$

Table 2.7 (cont.).

$$\frac{d\alpha_1}{ds} = \frac{l_1}{\rho}, \quad \frac{d\beta_1}{ds} = \frac{m_1}{\rho}, \quad \frac{d\gamma_1}{ds} = \frac{n_1}{\rho}, \quad \frac{d\lambda_1}{ds} = -\frac{l_1}{\tau_r}, \quad \frac{d\mu_1}{ds} = -\frac{m_1}{\tau_r}$$

$$\frac{dV_1}{ds} = -\frac{n_1}{\tau_r}$$

$$\frac{dl_1}{ds} = \frac{\alpha_1}{\rho} + \frac{\lambda_1}{\tau_r}, \quad \frac{dm_1}{ds} = \frac{\beta_1}{\rho} + \frac{\mu_1}{\tau_r}, \quad \frac{dn_1}{ds} = -\frac{\gamma_1}{\rho} + \frac{V_1}{\tau_r} \qquad \text{(d)}$$

$$t_t = \cos(\bar{t}_{01}, t) = I, \quad t_n = \cos(\bar{t}_{01}, \bar{n}) = -\frac{ds}{\rho}, \quad t_b = \cos(\bar{t}_{01}, b) = 0$$

Tangential	Normal	Binormal
$n_{1t} = \cos(\bar{n}_{01}, \bar{t}) = -t_{1n} = \dfrac{ds}{\rho}$,	$n_{1n} = \cos(\bar{n}_{01}, \bar{n}) = I$,	$n_{1b} = \cos(\bar{n}_{01}, \bar{b}) = -\dfrac{ds}{\tau_r}$
$b_{1t} = \cos(\bar{b}_{01}, \bar{t}) = t_{1b} = 0$,	$b_{1n} = \cos(\bar{b}_{01}, \bar{n}) = -n_b = \dfrac{ds}{\tau_r}$,	$b_{1b} = \cos(\bar{b}_{01}, \bar{b}) = I$ (e)

At point A an equally or unequally distributed load is given by the resultant force P and the corresponding resultant moment by M which can be replaced by their projections with subscriptions, 1, 2 and 3. Angular rotations in the respective directions are represented by ψ with specific subscripts. T_t, T_n and T_b are (tensile or compressive) shearing forces (T_n, T_b) and twisting moment M_t and bending moments M_n and M_b. The sign conventions are given in Figures (a) to (c) and the directions of moments are represented by double arrows. For a non-uniform load distribution $\overline{w}(s)$ per unit length of the

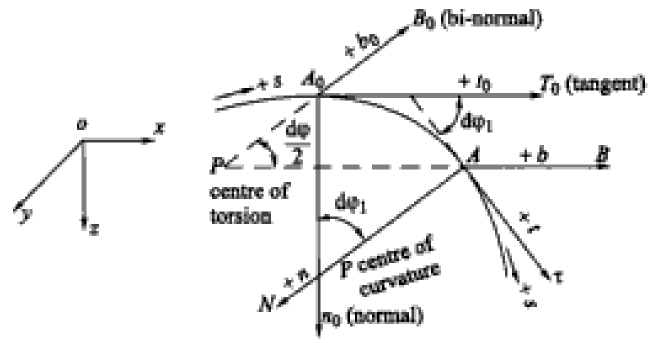


Figure (a). Geometry of the curve.

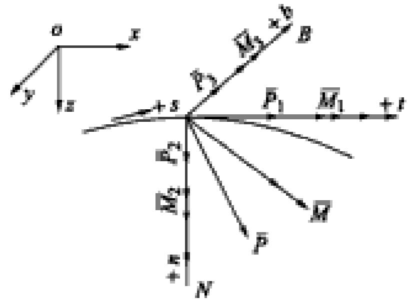


Figure (b). A twisted curved stair.

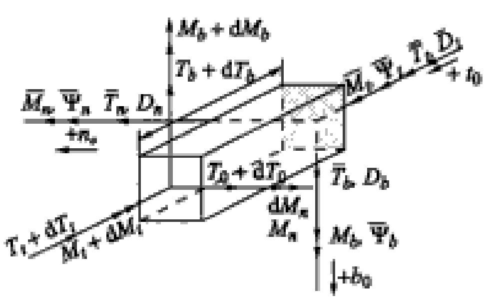


Figure (c). Moment, internal forces, displacements and rotations.

Table 2.7 (cont.).

stairs

$$\bar{w}(s) = w_t\bar{t} + w_n\bar{n} + w_b\bar{b} \tag{f}$$

and the non-uniform moment distribution of moment $\bar{m}(s)$

$$\bar{m}(s) = m_t\bar{t} + m_n\bar{n} + m_b\bar{b} \tag{g}$$

The following equilibrium equations have been derived by Cohen (14) with their application procedures:

A reference is made to Figures (e) to (g).

Equation of a helix about $xyz0$ Figure (e)

$$x = a\cos\Theta, \quad y = a\sin\Theta, \quad z = c\Theta, \quad C = \frac{H_1\phi}{2\pi} = \cot^{-1}\frac{C}{a} \tag{h}$$

$$ds = \sqrt{(dx^2 + dy^2 + dz^2)} \tag{i}$$

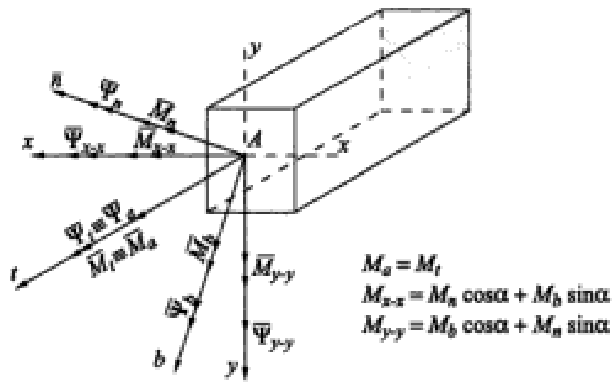


Figure (d). Sign convention.

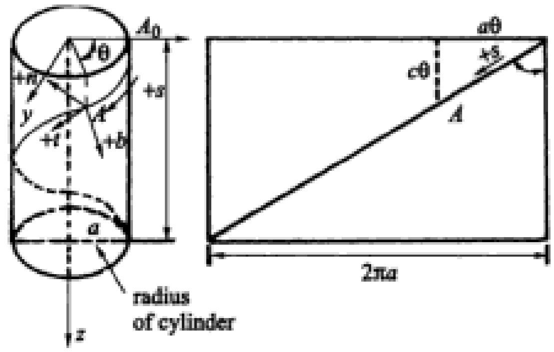


Figure (e). Cylindrical surface.

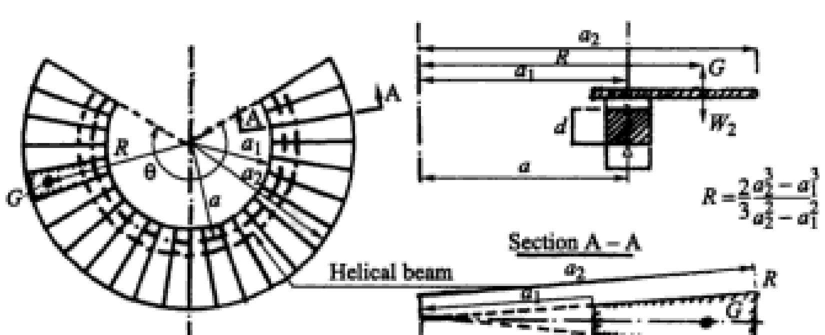


Figure (f). Plan of staircase.

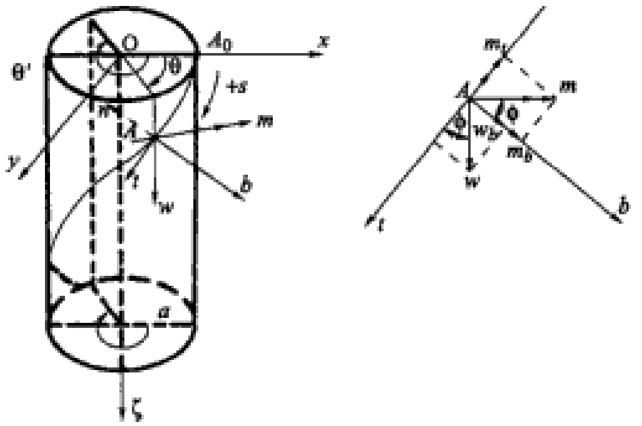


Figure (g).

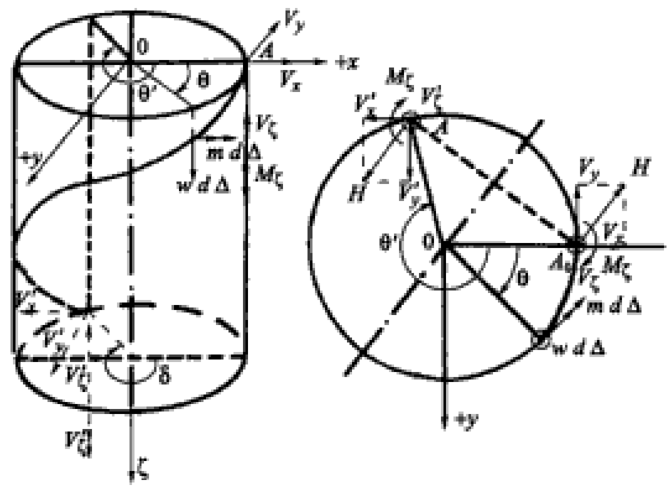


Figure (h).

$$s = \frac{\sqrt[4]{(c^2 + 1.\Theta)}}{a^2} = \frac{a}{\sin\phi}\Theta = \frac{\Theta}{K}$$

Where S is an independent variable, the coordinate axes and direction cosines are written as:

$$x = a\cos(Ks), \quad y = a\sin(Ks), \quad z = cKs \tag{j}$$

$$\alpha = -\sin\phi\sin(Ks), \quad \beta = \sin\phi\cos(Ks), \quad \gamma = \cos\phi$$

$$l_1 = -\cos(Ks), \quad m_1 = -\sin(Ks), \quad n_1 = 0$$

$$\lambda_1 = \cos\phi\sin(Ks), \quad \mu_1 = -\cos\phi\cos(Ks), \quad \nu_1 = \sin\phi$$

$$\rho = a/\sin^2\phi \text{ and is a constant}, \quad T_r = 2a/\sin^2\phi \text{ and is a constant} \tag{k}$$

Cohen took the uniformly distributed load and moment as Figure (h).

Changing S to Θ the new T'_s and M'_s are obtained

$$T_t = C_1 + C_2\sin\Theta + C_3\cos\Theta + aw\cot\phi\Theta$$

$$T_n = \frac{1}{\sin\phi}[C_2\cos\Theta - C_3\sin\Theta]$$

Table 2.7 (cont.).

$$T_b = -\cot\phi(C_2\cos\Theta - C_3\cos\Theta) + \tan\phi C_1 + aw\Theta$$
$$M_t = C_4 + C_5\sin\Theta + C_6\cos\Theta + a\cot\phi(C_2\cos\Theta - C_3\sin\Theta) + wa^2\Theta$$
$$M_n = \frac{1}{\sin\phi}[C_5\cos\Theta - C_6\sin\Theta] + a\cot\phi[C_2(\cos\Theta - \Theta\sin\Theta) - C_3(\sin\Theta + \Theta\cos\Theta)] + wa^2 + ma]$$

$$M_b = -\cot\phi(C_5\sin\Theta + C_6\cos\Theta) + \tan\phi C_4 + a\cot^2\phi\Theta(-C_2\cos\Theta + C_3\sin\Theta)$$
$$- \frac{a}{\sin^2\phi}(C_2\sin\Theta + C_3\cos\Theta) - \frac{a}{\cos^2\phi}C_1 - wa^2\cot\phi\Theta \quad \text{(l)}$$

where, C_1 to C_6 are constants for a determinate stair, there can be no moment about the axes Ox and Oy. Cohen derived the forces and moments at supports:

$$A_0 - V_x,\ V_y,\ V_z \text{ and } M_z$$

at

$$A - V'_x,\ V'_y,\ V'_z \text{ and } M'_z \quad \text{(m)}$$

The division of the vertical load between A_0 and A depends on the relative stiffness of the upper and lower supports. The final values are given below:

$$V_x = V'_x = -\frac{m}{\cos\phi}\cdot\left(-\frac{\sin\Theta'}{\Theta'}\right) + \frac{wa}{\cos\phi}\left(\frac{1+\cos\Theta'}{2} - \frac{\sin\Theta'}{\Theta'}\right)$$
$$V_y = V'_y = -\frac{m}{\cos\phi}\frac{1-\cos\Theta'}{\Theta'} + \frac{wa}{\cos\phi}\left(\frac{1-\cos\Theta'}{\Theta'} - \frac{\sin\Theta'}{2}\right)$$
$$V_x = V'_z = \frac{wa}{2\sin\phi}\Theta'$$
$$M_z = M'_z = \frac{ma}{\cos\phi}\cdot\left(-\frac{\cos\Theta'}{\Theta'}\right) + \frac{wa^2}{\cos\phi}\left(\frac{1-\cos\Theta'}{\Theta'} - \frac{\sin\Theta'}{\Theta'}\right)$$

At the origin of the helix A_0, the direction cosines are written as:

$$\alpha_1 = 0; \quad l_1 = -1; \quad \lambda = 0;$$
$$B_1 = \sin\phi; \quad m_1 = 0; \quad \mu_1 = \cos\phi \quad \text{(o)}$$
$$\gamma_1 = \cos\phi; \quad n_1 = 0; \quad V_1 = \sin\phi$$

when $\Theta = 0$,

$$T_{to} = \sin\phi V_y + \cos\phi V_z = C_1 + C_3$$
$$T_{no} = -V_x = \frac{C_2}{\sin\phi}$$
$$T_{bo} = -\cos\phi V_y + \sin\phi V_z = -\cot\phi C_3 + \tan\phi C_1$$
$$M_{to} = \cos\phi M_z = C_4 + C_6 \quad \text{(u)}$$
$$M_{no} = 0 = C_5 + a\cot\phi C_2 + wa^2 + ma$$
$$M_{bo} = \sin\phi M_z = -\cot\phi C_6 + \tan\phi C_4 - \frac{a}{\sin^2\phi}C_3 - \frac{a}{\cos^2\phi}C_1$$

By substituting in Equation (u) the values of V_x, V_y, V_z, M_z the values of the constants are as in Equation (v)

$$C_1 = -\frac{wa\Theta'}{2}\cot\phi$$
$$C_2 = \left(m_1\tan\phi_1\frac{\cos\Theta'}{\Theta'}\right) - wa\tan\phi\left[\frac{1-\cos\Theta'}{\Theta'} - \frac{\sin\Theta'}{2}\right] = C_2\tan\frac{\Theta'}{2} \quad \text{(v)}$$
$$C_4 = -\frac{wa^2\Theta'}{2}$$

Table 2.7 (cont.).

$$C_5 = -wa^2 - ma - a\cot\phi C_2$$
$$C_6 = m_1 a\frac{1-\cos\Theta'}{\Theta'} + wa^2\left(\frac{1-\cos\Theta'}{\Theta'} - \frac{\sin\Theta'}{2}\right) + \frac{wa^2\Theta'}{2} = -a\cot\Theta C_3 - C_4 \quad \text{(w)}$$

From Equation (l) and Equation (v) the shearing forces and bending moments about the principal axes and the normal force and twisting moments may be found for any point in a simply-supported helical stair. The values of T_{tA}, N_{nA}, T_{bA}, M_{tA}, M_{nA} and M_{bA} at the other end of the stair can be obtained by resolving the forces and calculating the moments about the three axes passing through A_0; they are:

$$T_{tA} = -T_{to}, \quad T_{nA} = T_{no}, \quad T_{bA} = -T_{bo}$$
$$M_{tA} = -M_{to}, \quad M_{nA} = M_{no} \quad \text{and} \quad M_{bA} = -M_{bo} \quad \text{(w)}$$

Statically indeterminate case (both ends fixed)

Here the equations of equilibrium are not sufficient and equations of deformation and angular rotation at any point are required. Cohen (1955) modified the general equations of equilibrium for a determinate case given in Section 2.7.3. The rotations are written as:

$$\begin{aligned} d\Psi_t &= \frac{ds}{\rho}\Psi_n + K_1\sigma M_t\, ds \\ d\Psi_n &= \frac{ds}{\rho}\Psi_t + \frac{ds}{\tau_r}\Psi_b + K_1 M_n\, ds \\ d\Psi_b &= ds + K_1 M_b\, ds \end{aligned} \tag{2.86}$$

where,

$$K_1 = \frac{1}{EI_n}, \quad K_2 = \frac{1}{EI_b} \quad \text{and} \quad \sigma' = \frac{I_n E}{IG}$$

J = polar moment of inertia $= \dfrac{A}{40 I_p}$, I_p = geometric polar moment of inertia.

Figure 2.13 shows displacements D_t, D_n, and D_b at A. The change of displacement when compared with that at A can be written as:

$$\frac{D_n}{\rho}ds - \frac{D_t}{\rho}ds + \frac{D_b}{\tau_r}ds - \frac{D_n}{\tau_r}ds \tag{2.87}$$

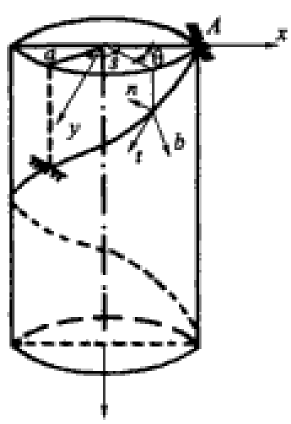


Figure 2.13. Helical staircase with all fixed ends.

The angular rotation (ψ_t, ψ_n, ψ_b) at A causes a displacement of A' of $C_1 \psi\, ds$ when the change of displacements caused by the external loads is added to these two displacements.

The total change of displacement is given by:

$$\text{a)} \quad dD_t = \frac{D_n}{\rho} ds + \frac{T_t}{EA} ds$$

$$\text{b)} \quad dD_n = -\frac{D_t}{\rho} ds + \frac{D_b}{\tau_r} ds + \Psi_b\, ds + \frac{T_n}{GA'n} ds \tag{2.88}$$

$$\text{c)} \quad dD_b = -\frac{D_n}{\rho} ds - \Psi_n\, ds + \frac{T_b}{GA'n} ds$$

The effects of the shearing and axial forces can be proved to be negligible; these are omitted from Equation (2.89)

$$\text{a)} \quad dD_t = \frac{D_n}{\rho} ds$$

$$\text{b)} \quad dD_n = -\frac{D_t}{\rho} ds + \frac{D_b}{\tau_r} ds + \Psi_b\, ds \tag{2.89}$$

$$\text{c)} \quad dD_b = -\frac{D_n}{\tau_r} ds - \Psi_n\, ds$$

Table 2.8 gives the brief set of equations which have been derived by Cohen (1955).

When both ends are fixed the staircase becomes six times indeterminate, there are six equations of equilibrium and twelve unknown reactions. The deformation equations are used to determine the unknown reactions. As shown in Figure 2.14 there is no displacement or angular

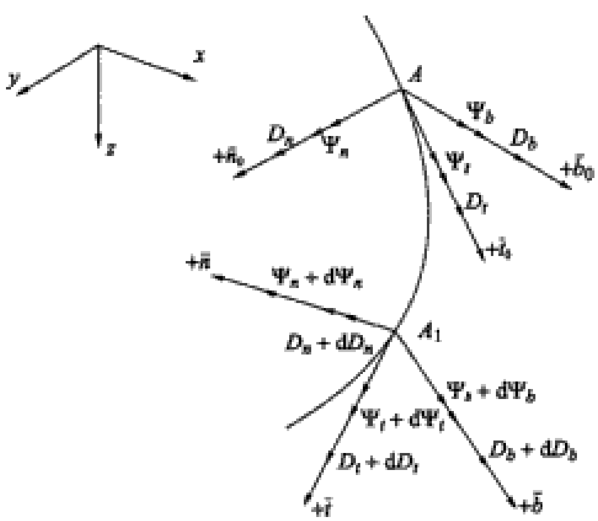


Figure 2.14. Displacement and other parameters for helical staircase with ends fixed.

rotation at A and B hence:

$$\Psi_{tA} = \Psi_{nA} = \Psi_{bA} = D_{tA} = D_{nA} = D_{bA} = 0$$
$$\Psi_{tB} = \Psi_{nB} = \Psi_{bB} = D_{tB} = D_{nB} = D_{bB} = 0 \tag{2.90}$$

For a circular helix inserting $\Theta = 0$ and $\Theta = \Theta'$ in Equations (a) and (c) Table 2.8 and inserting these in Equation (2.90) leads to Equations (f) of Table 2.9 using Equation (l) of Table 2.7.

Table 2.8. Equilibrium equations for indeterminate staircases.

Rotations:

$$\Psi_t = C_7 + C_8\sin\Theta + C_9\cos\Theta + A_1\Theta - A_2\Theta\frac{\sin\Theta}{2} - A_2\Theta\frac{\cos\Theta}{2} + A_4\frac{(-\Theta^2\sin\Theta - 3\Theta\cos\Theta)}{4} + A_5\frac{(-\Theta^2\cos\Theta - 3\Theta\sin\Theta)}{4} + A_6\frac{\Theta^2}{2}$$

$$\begin{aligned}\Psi_n = \frac{1}{\sin\phi}&\left[C_8\cos\Theta - C_9\sin\Theta + A_1 - A_2\Theta\frac{(\cos\Theta + \sin\Theta)}{2}\right]\\ &-\left[A_3\frac{(-\Theta\sin\Theta + \cos\Theta)}{2} + A_4\frac{(-\Theta^2\cos\Theta + \Theta\sin\Theta - 2\cos\Theta)}{2}\right]\\ &+\left[A_5\frac{(\Theta^2\sin\Theta + \Theta\cos\Theta + 3\sin\Theta)}{4} + A_6\Theta - \frac{K_1\sigma' a M_t}{\sin\phi}\right]\end{aligned} \tag{a}$$

$$\begin{aligned}\Psi_6 = \frac{1}{\sin\phi\cos\phi}&\left[-C_8\sin\Theta - C_9\cos\Theta - A_2\left(\frac{2\cos\Theta - \Theta\sin\Theta}{2}\right)\right]\\ &-\left[A_3\frac{(-2\sin\Theta - \Theta\cos\Theta)}{2} + A_4\frac{(\Theta^2\sin\Theta - \Theta\cos\Theta + 4\sin\Theta)}{4}\right.\\ &\left.+ A_5\frac{(\Theta^2\sin\Theta^2 + \Theta\sin\Theta + 4\cos\Theta)}{4} + A_6 + \Psi_t\sin^2\phi - K_1 a(1+\sigma')M_n\right]\end{aligned}$$

where,

$$A_1 = -\frac{K_2 a^2}{\sin^2\phi}C_1 + \frac{a}{\sin\phi}(K_1\sigma'\cos^2\phi + K_2\sin^2\phi)C_4$$

$$A_2 = -\frac{a^2\cos\phi}{\sin^2\phi}[2K_1(1+\sigma') + K_2]C_2 - \frac{a}{\sin\phi}[K_1(1+\sigma'\sin^2\phi) + K_2\cos^2\phi]C_5$$

$$A_3 = -\frac{a^2\cos\phi}{\sin^2\phi}[2K_1(1+\sigma') + K_2]C_3 - \frac{a}{\sin\phi}[K_1(1+\sigma'\sin^2\phi) + K_2\cos^2\phi]C_6$$

$$A_4 = +\frac{a^2\cos\phi}{\sin^2\phi}[K_1(1+\sigma'\sin^2\phi) + K_2\cos^2\phi]C_3$$

$$A_5 = -\frac{a^2\cos\phi}{\sin^2\phi}[K_1(1+\sigma'\sin^2\phi) + K_2\cos^2\phi]C_2$$

$$A_6 = +\frac{wa^3\cos'^2\phi}{\sin\phi}(K_1\sigma' - K_2)$$

Displacements:

A reference is made to Table 2.9.

Consider the staircase shown in Figure (f) as having a uniformly distributed load of Sw kN per metre and a uniformly-distributed bending moment of m_1 kN m, of the helix, for which the values of T_t, T_n, T_b, M_t, M_n and M_b are given by Equations (l) of Table 2.7. The following displacement parameters are obtained by Cohen.

Table 2.9. Equilibrium equations for indeterminate staircases (cont.).

$$
\begin{aligned}
D_t = {} & C_{10} + C_{11}\sin\Theta + C_{12}\cos\Theta + B_1\Theta - B_2\frac{\Theta\sin\Theta}{2} - B_3\frac{\Theta\cos\Theta}{2} \\
& + B_4\left(\frac{-\Theta^2\sin\Theta - 3\Theta\cos\Theta}{4}\right) + B_5\left(\frac{-\Theta^2\cos\Theta + 3\Theta\sin\Theta}{4}\right) + B_6\frac{\Theta^2}{2} \\
& + B_7\left(\frac{-\Theta^3\cos\Theta}{6} + \frac{3\Theta^2\sin\Theta}{4} + \frac{7\Theta^2\cos\Theta}{4}\right) + B_8\left(\frac{-\Theta^3\sin\Theta}{6} - \frac{3\Theta^3\cos\Theta}{4} + \frac{7\Theta\sin\Theta}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
D_n = {} & \frac{1}{\sin\phi}\Bigg[C_{11}\cos\Theta - C_{12}\sin\Theta + B_1 - B_2\left(\frac{\Theta\cos\Theta + \sin\Theta}{2}\right) \\
& - B_3\left(\frac{-\Theta\sin\Theta + \cos\Theta}{2}\right) + B_4\left(\frac{-\Theta^2\cos\Theta + \Theta\sin\Theta - 3\cos\Theta}{4}\right) \\
& + B_5\left(\frac{\Theta^2\sin^2\Theta + \Theta\cos\Theta + 3\sin\Theta}{4}\right) + B_6\Theta \\
& + B_7\left(\frac{\Theta^3\sin\Theta}{6} + \frac{\Theta^2\cos\Theta}{4} - \frac{\Theta\sin\Theta}{4} + \frac{7\cos\Theta}{4}\right) \\
& + B_8\left(\frac{-\Theta^3\cos\Theta}{6} + \frac{\Theta^2\sin\Theta}{4} - \frac{\Theta\cos\Theta}{4} + \frac{7\sin\Theta}{4}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
D_b = {} & \frac{1}{\sin\phi\cos\phi}\Bigg[-C_{11}\sin\Theta - C_{12}\cos\Theta - B_2\left(\frac{-\Theta\sin\Theta + 2\cos\Theta}{2}\right) \\
& - B_3\left(\frac{-\Theta\cos\Theta - 2\sin\Theta}{2}\right) + B_4\left(\frac{\Theta^2\sin\Theta + \Theta\cos\Theta + 4\sin\Theta}{4}\right) \\
& + B_5\left(\frac{\Theta^2\cos\Theta + \Theta\sin\Theta + 4\cos\Theta}{4}\right) + B_6 \\
& + B_7\left(\frac{\Theta^3\cos\Theta}{6} + \frac{\Theta^2\sin\Theta}{4} + \frac{\Theta\cos\Theta}{4} - 2\sin\Theta\right) \\
& + B_8\left(\frac{\Theta^3\sin\Theta}{6} - \frac{\Theta^2\cos\Theta}{4} + \frac{\Theta\sin\Theta}{4} + 2\cos\Theta\right) + \sin^2\Theta D_t - a\Psi_b\Bigg]
\end{aligned}
\tag{c}
$$

where constants B_1 to B_6 are given by

$$
\frac{d^3 D_t}{d\Theta} + \frac{dD_t}{d\Theta} = B_1 + B_2\sin\Theta + B_3\cos\Theta + B_4\Theta\sin\Theta + B_5\Theta\cos\Theta + B_6\Theta + B_7\Theta^2\cos\Theta + B_8\Theta^2\sin\Theta \tag{d}
$$

Table 2.9 (cont.).

in which,

$$B_1 = \frac{a}{\sin\phi\cos\phi}\left[-A_1\cos 2\phi + \frac{K_1\sigma' a\cos^2\phi}{\sin\phi}C_4\right]$$

$$B_2 = \frac{a}{\sin\phi\cos\phi}\left[(1+\cos 2\phi)(C_9 - 0.75A_5) + \left(\frac{3+\cos 2\phi}{2}\right)A_2 + \frac{K_1 a}{\sin\phi}(1+\sigma'+\sigma'\cos\phi)C_{52}\right]$$
$$+ \frac{2a^2K_1(1+\sigma')\cos\phi}{\sin^2\phi} \times C_2$$

$$B_3 = \hat{L}\left[\widehat{M}(-C_8 + 0.75A_4) + \left(\frac{3+\cos 2\phi}{2}\right)A_3 + \frac{K_1 a}{\sin\phi}(\widehat{N})C_6 + \frac{2a^2K_1(1+\sigma')\cos\phi}{\sin^2\phi}C_3\right]$$

where,

$$\hat{L} = \frac{a}{\sin\phi\cos\phi}, \quad \widehat{N} = (1+\sigma'+\sigma'\cos\phi)$$

$$B_4 = \hat{L}\left[-\widehat{O}A_3 + \widehat{S}A_4 - \frac{a^2K_1\cos\phi}{\sin^2\phi}\widehat{N}C_3\right] \qquad \text{(e)}$$

where,

$$\widehat{O} = \left(\frac{1+\cos 2\phi}{2}\right); \quad \widehat{S} = \left(\frac{3-\cos 2\phi}{4}\right)$$

$$B_5 = \hat{L}\left[\frac{\widehat{O}}{2}A_2 + \widehat{S}A_5 + \frac{a^2K\cos\phi}{\sin^2\phi}\widehat{N}C_2\right]$$

$$B_6 = \hat{L}\left[-A_6\cos 2\phi + \frac{wa^2K_1\sigma'\cos^2\phi}{\sin\phi}\right]$$

$$B_7 = \hat{L}[\widehat{O}A_4] \quad \text{and} \quad B_8 = \hat{L}[\widehat{O}A_5]$$

M Values:

$$M_{tA} = C_4 + C_6$$

$$M_{tB} = C_4 + C_5\sin\Theta' + C_6\cos\Theta' + a\Theta'\cot\Theta(C_2\cos\Theta' - C_3\sin\Theta') + wa^2\Theta'$$

$$M_{nA} = \frac{1}{\sin\phi}(C_5 + a\cot\phi C_2 + wa^2 + ma)$$

$$M_{nB} = \frac{1}{\sin\phi}\left[C_5\cos\Theta' - C_6\sin\Theta' + a\cot\phi\{C_2(\cos\Theta' - \Theta'\sin\Theta') - C_3(\sin\Theta' + \Theta'\cos\Theta')\}\right.$$
$$\left. + wa^2 + ma\right]$$

$$\epsilon = \frac{K_2}{K_1}, \quad \epsilon_1 = \sigma'\cos^2\phi + \varepsilon\sin^2\phi, \quad \epsilon_2 = 2(1+\sigma') + \varepsilon \qquad \text{(f)}$$

$$\varepsilon_3 = 1 + \sigma'\sin^2\phi + \varepsilon\cos^2\phi$$

$$\varepsilon_4 = (\sigma' - \epsilon)\sin\phi\cos\phi$$

In the special case of symmetrical loading, the solution may be simplified considerably.

$$C_1 = -\frac{wa}{2}\Theta'\cot\phi, \quad C_2 = \frac{-C_5 - \sin\Theta' - C_6(1+\cos\Theta')}{a\Theta'\cot\phi} = -C_3\cot\frac{\Theta'}{2}$$

$$C_3 = \frac{C_5(1-\cos\Theta') + C_6\sin\Theta'}{a\Theta'\cot\phi} = -C_2\tan\frac{\Theta'}{2}$$

$$C_4 = -\frac{wa^2\Theta}{2}$$

Table 2.9 (cont.).

a) $C_1 + C_3 = -C_1 - C_2 \sin\Theta' - C_3 \cos\Theta' - aw\Theta' \cot\phi$

b) $C_2 = C_2 \cos\Theta' - C_3 \sin\Theta'$

c) $-\cot\phi C_3 + \tan\phi C_1 = \cot\phi(C_2 \sin\Theta' + C_3\cos\Theta') - \tan\phi C_1 - aw\Theta'$

d) $C_4 + C_6 = -C_4 - C_5 \sin\Theta' - C_6\cos\Theta' - a\Theta'\cot\phi(C_2\cos\Theta' - C_3\sin\Theta') - wa^2\Theta'$

e) $C_5 + a\cot\phi C_2 + wa^2 + ma = C_5\cos\Theta' - C_6\sin\Theta' + a\cot\phi\{C_2(\cos\Theta' - \Theta'\sin\Theta')$ (g)

$\quad - C_3(\sin\Theta' + \Theta'\cos\Theta')\} + wa^2 + ma$

f) $-\cot\phi C_6 + \tan\phi C_4 - \dfrac{a}{\sin^2\phi}C_3 - \dfrac{a}{\cos^2\phi}C_1 = \cot\phi(C_5\sin\Theta' + C_6\cos\Theta') - \tan\phi C_4$

$\quad - a\Theta'\cot^2\phi(C_2\cos\Theta' + C_3\sin\Theta') + \dfrac{a}{\sin^2\phi}(C_2\sin\Theta' + C_3\cos\Theta') + \dfrac{a}{\cos^2\phi}C_1 + wa^2\delta\cot\phi$

The remaining two constants are determined by Cohen (1955) as:

$$d_1C_5 + e_1C_6 = f_1; \quad d_2C_5 + e_2C_6 = f_2 \tag{h}$$

in which,

$$d_1 = \frac{\sin^2\Theta'}{2}\varepsilon_1\sin^2\phi - \frac{\Theta'\sin 2\Theta'}{8}\left(\cos^2\phi - \varepsilon + \varepsilon_1\sin^2\phi\right) - \frac{\Theta'}{4}\cos^2\phi(\varepsilon_3 - 2\varepsilon - 2)$$
$$+ \frac{\Theta'\sin\Theta'}{2}\sin^2\phi(\varepsilon_1 - 2\varepsilon) - \frac{\Theta'^2\cos\Theta'}{2}\cos^2\phi(\varepsilon_3 - 1) - \frac{\Theta'^3\sin\Theta'}{6}\varepsilon_3\cos^2\phi$$

$$e_1 = (\sin\Theta' - \Theta'\cos\Theta')(1 + \cos\phi)\frac{\varepsilon_1\sin^2\phi}{2} + \frac{\Theta'^2\sin\Theta'}{2}\cos^2\phi$$
$$+ \frac{\Theta'\sin^2\Theta'}{4}(\cos^2\phi - \varepsilon - \varepsilon_1\sin^2\phi) + \frac{\Theta'^3\varepsilon_3\cos^2\phi}{12}(1 - 2\cos\Theta')$$

$$f_1 = -wa^2\Theta'\cos^2\phi\left[\Theta'\cos\Theta'\left(2\varepsilon_3 - \frac{\varepsilon}{2} - 1\right) + \Theta'\left(\varepsilon_3 - \frac{\varepsilon}{2} - 1\right) + \frac{\Theta'^2\sin\Theta'}{2}(\varepsilon_3 - 1)\right.$$
$$\left. + \sin\Theta'(-3\varepsilon_3 + \varepsilon + 2)\right] - ma\Theta'\cos^2\phi(1 + \delta')(\Theta'\cos\Theta' - \sin\Theta')$$

$$d_2 = \Theta'^2\sin\Theta'\frac{\cos^2\phi}{4}(\varepsilon_3 - 2) - \frac{\Theta'^3\varepsilon_3\cos^2\phi}{12}(1 + 2\cos\Theta')$$
$$+ (\cos^2\phi - \varepsilon)\left(\frac{1 - \cos\Theta}{4}\right)(\Theta'\cos\Theta' - \sin\Theta') + \Theta'(1 - \cos\Theta')\left(\frac{\varepsilon + \cos^2\phi}{2}\right)$$

$$e_2 = \frac{\Theta'\cos^2\phi}{2}(\varepsilon + 1) - \frac{\Theta'^2\cos\Theta'}{4}\cos^2\phi(\varepsilon_3 - 2) + \frac{\Theta'\sin\Theta'}{4}\left(2\varepsilon\sin^2\phi + 2\cos^2\phi + \frac{\varepsilon}{3}\cos^2\phi\right)$$
$$+ (\cos^2\phi - \varepsilon)\frac{\sin\Theta'}{4}(\Theta'\cos\Theta' + \sin\Theta') + \varepsilon_1\sin^2\phi\frac{\sin\Theta'}{4}(\Theta'\cos\Theta' - \sin\Theta') + \varepsilon_3\Theta'^3\sin\Theta'\frac{\cos^2\phi}{6}$$

$$f_2 = -wa^2\Theta'\cos^2\phi\left[\frac{\Theta'\sin\Theta'}{2}(3\varepsilon_3 - \varepsilon - 1) - \frac{\Theta'^2\cos\Theta'}{2}(\varepsilon_3 - 1)\right.$$
$$\left. - (1 - \cos\Theta')(2\varepsilon_3 - \varepsilon - 2)\right] + ma\Theta'\cos^2\phi(1 + \delta')\Theta'^2\sin\Theta'$$

Hence

$$C_5 = \frac{f_1e_2 - f_2e_1}{d_1e_2 - d_2e_1}; \quad C_6 = \frac{d_1f_2 - d_2f_1}{d_1e_2 - d_2e_1} \tag{i}$$

EXAMPLE 2.8

Assuming the staircase given in Example 2.7 is now fixed at the far ends and also assuming that the loads and other parameters are kept the same, re-analyse the stairs for the unknowns T_t, T_n, M_t, M_n and M_b and constants C_s.

SOLUTION

A helical staircase with fixed ends

$$\varepsilon = \frac{K_1}{K_2} = \frac{I_n}{I_b} = 0.397$$

$$\sigma' = \frac{I_n}{J} \cdot \frac{E}{G} = 0.957$$

$$\varepsilon_1 = 0.654$$

$$\varepsilon_2 = 4.3, \quad \varepsilon_3 = 1{,}70 \quad \text{and} \quad \varepsilon_4 = 0.279$$

$$d_1 = 12.0, \qquad d_2 = 3.0$$

$$e_1 = 5.96, \quad e_2 = -4.75$$

$$f_1 = 34425, \quad f_2 = -37910$$

$$C_1 = -8.7789456, \qquad C_2 = 3.6460103$$

$$C_3 = -0.1333309, \quad C_4 = -8.3487387$$

$$C_5 = -0.2440182, \qquad C_6 = 6.9215908$$

Substituting into Equation (I) of Table 2.6 forces and moments are obtained and can be found in Table 2.10.

Table 2.10. Summary of results.

Parameters	Θ'								
	0	$\pi a/6$	$\pi a/3$	$\pi a/2$	$2\pi a/3$	$5\pi a/6$	$6\pi a/6$	$7\pi a/6$	$4\pi a/3$
T_t (kN)	−11.854	−10.120	−7.459	−3.959	0	3.959	7.459	10.120	11.854
T_n (kN)	−2.384	0	2.384	4.128	4.768	4.128	2.384	0	−2.384
T_b (kN)	−6.774	−3.950	−1.993	−0.778	0	0.778	1.993	3.950	6.774
M_t (kN m)	0.1154πa 1.592	+1.851 1.790	1.329	0.665	0	−0.665	−1.329	−1.790 −1.851πa	0.1154πa −1.592
M_n (kN m)	3.167	0.488	−0.529	−0.644	−0.603	−0.644	0.529	0.488	3.167
M_b (kN m)	0.202πa 3.523	4.120 4.087	3.751	2.251	0	−2.251	−3.751	−4.087 −4.120πa	0.2023πa −3.523

CHAPTER 3

Structural analysis of staircases: Modern methods

3.1 INTRODUCTION

This chapter covers the analysis of stairs using three different methods. They are:

– Flexibility Method
– Stiffness Method
– Finite Element Method

3.2 FLEXIBILITY METHOD

The flexibility method of a staircase is defined as its displacement caused by a unit force. The displacement is derived using the strain energy method.

The total strain energy is given by (e.g. for bending)

$$U = \int \frac{M^2\,\mathrm{d}s}{2EI} \tag{3.1}$$

where U = total strain energy; M = bending moment; E = Young's modulus; I = second moment of inertia.

Table 3.1 gives the general sign convention. The following steps are taken into consideration:

– Establish static indeterminacy number.

– Choose release system to reduce structure of the stair to statically determinate.

– This may be done so either by removing supports which are causing indeterminacy or making individual spans determinate by inserting artificial hinges at supports with 'bi-actions'.

– Draw B. M. diagrams due to external loads on the determinate structure. It is termed as 'm_0' diagram.

– Remove external loads and apply unit load or unit 'bi-action' at each of the releases in turn. These are called m_1 to m_n diagrams defining corresponding 'flexibility coefficients' such as f_{11} to f_{nn}. The releases of the flexibility coefficients 'f', are X_1 to X_n. The value of M given in Eq. (3.1) is, by superposition, given.

$$M = m_0 + m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \tag{3.2}$$

Table 3.1. Sign convention.

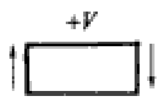
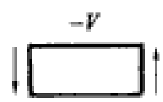
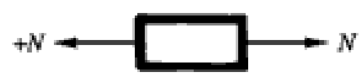
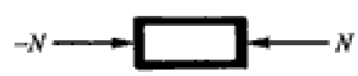
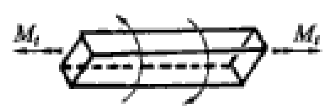
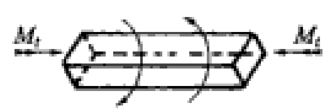
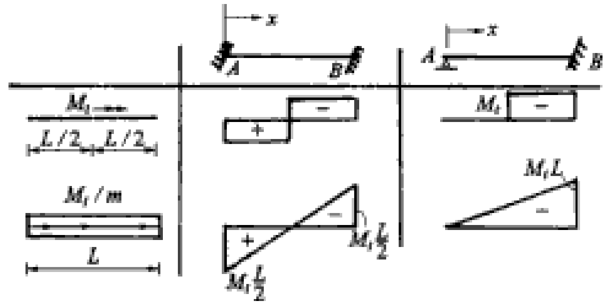
Bending Moment Bending moment: positive (Fig. (a)) If the tension is at the bottom, and the shape is concave, bending moment is positive.	 a) Concave upward
Bending moment: negative If the tension is at the top and the shape is convex, bending moment is negative	 b) Convex upward
Shear Force Shear force: positive If the force goes upward at the left side of the element and downwards at the right hand side.	 c) Leftside of the element (force up)
Shear force: negative The opposite to conversion adopted for the negative shear	 d) Leftside of the element (force down)
Axial force or normal force N When the beam is stretched the force is positive and when it is compressed the force is negative.	 e) Axial force  f) Normal force
Torsion	 g) Torsional moment (positive)  h) Torsional moment (negative)

Table 3.1 (cont.).

When the torsional moment is applied (Fig. (g)) with the vector extending outwards it is treated as positive. When it is occurring in the opposite direction, it is called negative. In order to illustrate further, a beam with two different boundary conditions is examined. A reference is made to Fig. (i).



i) A beam with two different boundary conditions

The final bending moment diagram M is drawn indicating principal values. Hence f_{ij} flexibility coefficients are written as

$$f_{ij}^{m} = \int \frac{m_i m_j}{EI}\,\mathrm{d}s \tag{3.2a}$$

For example, for a stair with two indeterminacies with reference coordinates $\curvearrowright_1$ and $\curvearrowright_2$ the following relation can be written

$$\begin{aligned} f_{11}X_1 + f_{12}X_2 &= -\delta_{10} \\ f_{21}X_1 + f_{22}X_2 &= -\delta_{20} \end{aligned} \tag{3.3}$$

In a matrix form, Eq. (3.3) is written as

$$\begin{aligned} &\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = -\begin{Bmatrix} \delta_{10} \\ \delta_{20} \end{Bmatrix} \\ &[f]\{X\} = -\{\delta_{10}\} \end{aligned} \tag{3.4}$$

By inverting the flexibility matrix, the indeterminacy X can be computed as

$$\{X\} = [f]^{-1}\{-\delta_{i0}\} \tag{3.5}$$

The values of $\{X\}$ from Eq. (3.5) are substituted into Eq. (3.2) for various ordinates of the final bending moment diagram M.

If the staircase components are subjected to shear, axial and torsional effects, the above method is repeated and Eq. (3.2) can be written as:

$$\begin{aligned} V &= \text{Shear} = v_0 + v_1X_1 + v_2X_2 + \cdots + v_jX_j + \cdots + v_nX_n \\ N &= \text{Axial} = n_0 + n_1X_1 + n_2X_2 + \cdots + n_jX_j + \cdots + n_nX_n \\ T &= \text{Torsion} = T_0 + T_1X_1 + T_2X_2 + \cdots + T_jX_j + \cdots + T_nX_n \end{aligned} \tag{3.6}$$

The total strain energy of a loaded stair is given by

$$U = \int^{s^*} \frac{M^2\,\mathrm{d}s}{2EI} + \int^{s^*} \frac{N^2\,\mathrm{d}s}{2EA} + K'\int^{s^*} \frac{V^2\,\mathrm{d}s}{2GA} + \int^{s^*} \frac{T^2\,\mathrm{d}s}{2GJ} \tag{3.7}$$

where M = bending moment; N = axial force, V = shear, T = torsion, A = cross-section, K' = shape constant, G = rigidity or shear modulus, J = polar moment of inertia, s^* = stair structure.

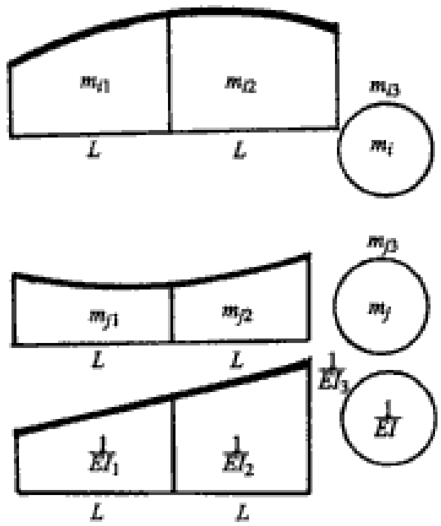
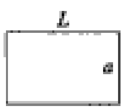
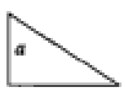
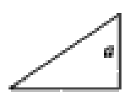
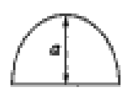
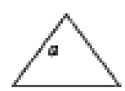
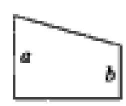
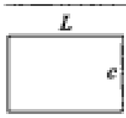
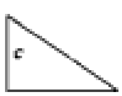
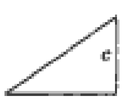
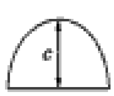
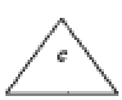
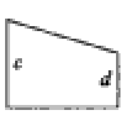


Figure 3.1. Simpson's Rule I.

Table 3.2. Product integrals $\int_0^L m_i m_j \, dx$.

m_i / m_j						
	Lac	$\frac{1}{2}Lac$	$\frac{1}{2}Lac$	$\frac{2}{3}Lac$	$\frac{1}{2}Lac$	$\frac{1}{2}L(a+b)c$
	$\frac{1}{2}Lac$	$\frac{1}{3}Lac$	$\frac{1}{6}Lac$	$\frac{1}{3}Lac$	$\frac{1}{4}Lac$	$\frac{1}{6}L(2a+b)c$
	$\frac{1}{2}Lac$	$\frac{1}{6}Lac$	$\frac{1}{3}Lac$	$\frac{1}{3}Lac$	$\frac{1}{4}Lac$	$\frac{1}{6}L(a+2b)c$
	$\frac{2}{3}Lac$	$\frac{1}{3}Lac$	$\frac{1}{3}Lac$	$\frac{8}{15}Lac$	$\frac{5}{12}Lac$	$\frac{1}{3}L(a+b)c$
	$\frac{1}{2}Lac$	$\frac{1}{4}Lac$	$\frac{1}{4}Lac$	$\frac{5}{12}Lac$	$\frac{5}{15}Lac$	$\frac{1}{4}L(a+b)c$
	$\frac{1}{2}La(c+d)$	$\frac{1}{6}La(2c+d)$	$\frac{1}{6}La(c+2d)$	$\frac{1}{3}La(c+d)$	$\frac{1}{4}La(c+d)$	$\frac{1}{6}La(2c+d)+b(2d+c)$

The corresponding flexibility coefficients are written as

$$f_{ij} = \int^{m} \frac{m_i m_j}{EI}\,\mathrm{d}s + \int^{n} \frac{n_i n_0}{EA}\,\mathrm{d}s + K' \int^{V} \frac{v_i v_0}{GA}\,\mathrm{d}s + \int^{T} \frac{T_i T_0}{GJ}\,\mathrm{d}s \qquad (3.8)$$

$$f_{ij} = f_{ij}^{m} + f_{ij}^{n} + f_{ij}^{V} + f_{ij}^{T}$$

The combined values of the above, where loadings are involved, are given as

$$\partial_{0j} = \int^{m} \frac{m_i m_0}{EI}\,\mathrm{d}s + \int^{n} \frac{n_i n_0}{EA}\,\mathrm{d}s + K' \int^{V} \frac{v_i v_0}{GA}\,\mathrm{d}s + \int^{T} \frac{T_i T_0}{GJ}\,\mathrm{d}s \qquad (3.9)$$

$$\partial_{i0} = \partial_{i0}^{m} + \partial_{i0}^{n} + \partial_{i0}^{V} + \partial_{i0}^{T}$$

The best and a popular method to solve intergrals in the above equations is by the Simpson's Rule. Each shape is divided into equal spaces and contained by three ordinates. Two methods are adopted

$$\int m_i\,\mathrm{d}s = \frac{L}{3}(m_{i1} + m_{i2} + m_{i3}) \qquad (3.9a)$$

$$\int \frac{m_i m_j}{EI}\,\mathrm{d}s = \frac{L}{3}\left[\frac{m_{i1} m_{j1}}{EI_1} + 4\frac{m_{i2} m_{j2}}{EI_2} + \frac{m_{i3} m_{j3}}{EI_3}\right] \qquad (3.9b)$$

The total is L under each curve.

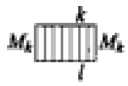
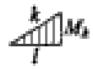
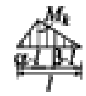
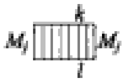
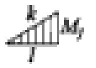
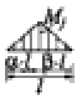
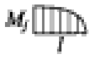
In Eq. (3.9a); $L/3$ is changed to $L/6$ outside the bracket.

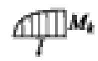
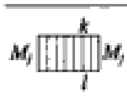
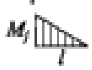
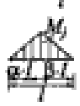
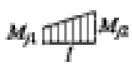
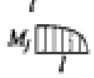
In order to minimise the number of calculations, flexibility coefficients are tabulated using Simpson's Rule for various shapes. Table 3.2 gives various such values against noted shapes. Tables 3.2 and 3.3 demonstrate the use of the flexibility method on staircases with different combinations of bending, shear, axial and torsional effects. Example 3.1, by using these tables, sets out step by step calculations for a free standing single flight stair.

3.2.1 *Wedge beam analysis*

A wedge beam at the top of the flight can be subjected to different loads at different positions. Table 3.4 shows some of the cases. It is important to evaluate deflections and rotations. These can then easily be incorporated into the main stairs as external effects. In a way they can act as special boundary conditions. It is also possible that due to built-in floors and beams, the wedge beam may be subjected to a moment, axial thrust and shear. Using the flexibility method, the combined effect of these three can be incorporated into a single matrix and the final results, such as rotations and displacements can then be achieved. Table 3.5 gives a step by step method of achieving such results.

Table 3.3. Flexibility chart: integral $\int M_i M_k \, ds$.

			
	$l M_i M_k$	$\frac{1}{2} l M_i M_k$	$\frac{1}{2} l M_i M_k$
	$\frac{1}{2} l M_i M_k$	$\frac{1}{3} l M_i M_k$	$\frac{1}{6} l(1+\alpha) M_i M_k$
	$\frac{1}{2} l M_i M_k$	$\frac{1}{6} l M_i M_k$	$\frac{1}{6} l(1+\beta) M_i M_k$
	$\frac{1}{2} l M_i M_k$	$\frac{1}{6} l(1+\alpha) M_i M_k$	$\frac{1}{3} l M_i M_k$
	$\frac{1}{2} l(M_{i_1} + M_{i_2}) M_k$	$\frac{1}{6} l(M_{i_1} + 2M_{i_2}) M_k$	$\frac{1}{6} l M_k[(1+\beta) M_{i_1} + (1+\alpha) M_{i_2}]$
	$\frac{2}{3} l M_i M_k$	$\frac{5}{12} l M_i M_k$	$\frac{1}{12} l(5-\beta-\beta^2) M_i$
	$\frac{2}{3} l M_i M_k$	$\frac{1}{4} l M_i M_k$	$\frac{1}{12} l(5-\alpha-\alpha^2) M_i$
	$\frac{1}{3} l M_i M_k$	$\frac{1}{4} l M_i M_k$	$\frac{1}{12} l(1+\alpha+\alpha^2) M_i$
	$\frac{1}{3} l M_i M_k$	$\frac{1}{12} l M_i M_k$	$\frac{1}{12} l(1+\beta+\beta^2) M_i$
$\int M_k M_k \, ds$	$l M_k M_k$	$\frac{1}{3} l M_k M_k$	$\frac{1}{3} l M_k M_k$

				
	$\frac{1}{2} l M_i (M_{k1} + M_{k2})$	$\frac{2}{3} l M_i M_k$	$\frac{2}{3} l M_i M_k$	$\frac{1}{3} l M_i M_k$
	$\frac{1}{6} l M_i (M_{k1} + 2M_{k2})$	$\frac{1}{3} l M_i M_k$	$\frac{5}{12} l M_i M_k$	$\frac{1}{4} l M_i M_k$
	$\frac{1}{6} l M_i (2M_{k1} + M_{k2})$	$\frac{1}{3} l M_i M_k$	$\frac{1}{4} l M_i M_k$	$\frac{1}{12} l M_i M_k$
	$\frac{1}{6} l M_i [(1+\beta) M_{k1} + (1+\alpha) M_{k2}]$	$\frac{1}{3} l(1+\alpha\beta) M_i M_k$	$\frac{1}{12} l(5-\beta-\beta^2) M_i M_k$	$\frac{1}{12} l(1+\alpha+\alpha^2) M_i M_k$
	$\frac{1}{6} l(2M_{i1} M_{k1} + M_{i1} M_{k2} + M_{i2} M_{k1} + 2M_{i2} M_{k2})$	$\frac{1}{3} l(M_{i1} + M_{i2}) M_k$	$\frac{1}{12} l(3M_{i1} + 5M_{i2}) M_k$	$\frac{1}{12} l(M_{i1} + 3M_{i2}) M_k$
	$\frac{1}{12} l M_i (3M_{k1} + 5M_{k2})$	$\frac{7}{15} l M_i M_k$	$\frac{8}{15} l M_i M_k$	$\frac{3}{10} l M_i M_k$
	$\frac{1}{12} l M_i (5M_{k1} + 3M_{k2})$	$\frac{7}{15} l M_i M_k$	$\frac{11}{30} l M_i M_k$	$\frac{2}{15} l M_i M_k$
	$\frac{1}{12} l M_i (M_{k1} + 3M_{k2})$	$\frac{1}{5} l M_i M_k$	$\frac{3}{10} l M_i M_k$	$\frac{1}{5} l M_i M_k$
	$\frac{1}{12} l M_i (3M_{k1} + M_{k2})$	$\frac{1}{5} l M_i M_k$	$\frac{2}{15} l M_i M_k$	$\frac{1}{30} l M_i M_k$
$\int M_k M_k \, ds$	$\frac{1}{3} l(M_{k1}^2 + M_{k2}^2 + M_{k1} M_{k2})$	$\frac{8}{15} l M_k M_k$	$\frac{8}{15} l M_k M_k$	$\frac{1}{5} l M_k M_k$

Note: M can be taken as m; M_k can be taken as M_j; $s = l$.

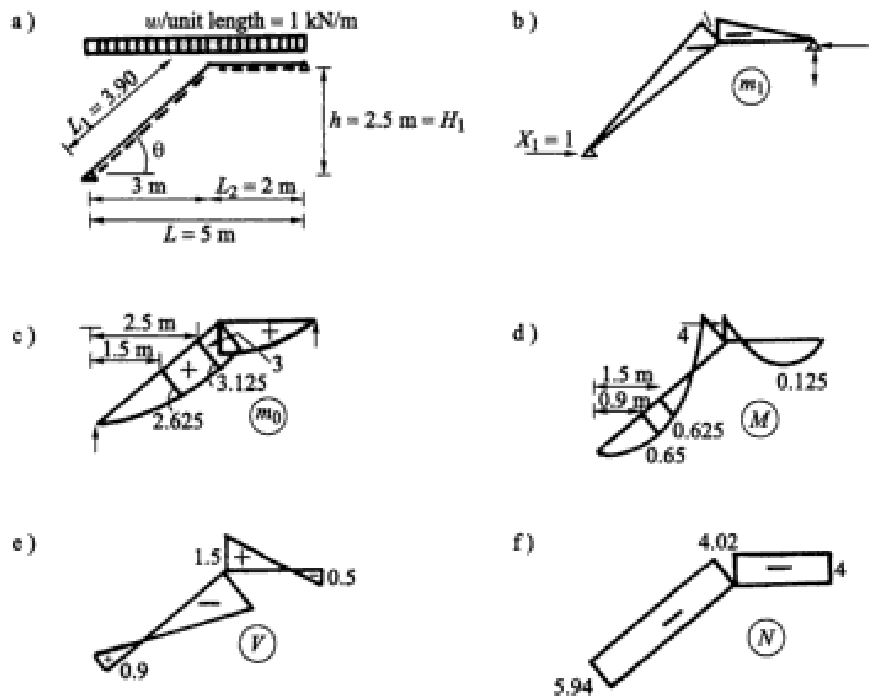


Figure 3.2. Flexibility diagram for Case (i).

EXAMPLE 3.1: *Free standing single flight stairs*

A free standing single flight staircase ACB (Fig. 3.2) is to be analyzed using the following loading and boundary conditions:

(i) Simply supported or pinned at A and B and rigidly jointed at C. A load of 1 kN/m is placed vertically on the plane projection of ACB, i.e. on both the stair and the landing.

(ii) Boundary conditions are the same as in (i) but a load of 1 kN/m is placed on the landing CB.

(iii) Supports A and B are fixed and a load of 1 kN/m is placed on the landing and the span and the height are taken as 8 m and 3 m, respectively. Ignore torsional effects.

SOLUTION

A Single Flight with a top landing:

EI constant

Case (i) release support A and is replaced by:

$X_1 = 1$ as a horizontal force

Take moment about B

$$R_A L = 1 \times h$$

or

$$R_A = \frac{h}{L} = \frac{2.5}{5} = 0.5$$

$$H_A = -1$$

$$M_{CA} = -1 \times 2.5 + 0.5 \times 3 = -1 = m_{ca}$$

$$m_0 = 1 \times \frac{5^2}{8} = 3.125 \text{ kN m}$$

Assuming the slope is at an angle θ

$$\tan\theta = \frac{2.5}{3} = 0.833$$

$$\theta = 39.8^\circ$$

$$L_1 = \frac{2.5}{\sin\theta} = \frac{2.5}{0.6401} = 3.90 \text{ m}$$

$$\sin\theta = 0.6401, \quad \cos\theta = 0.7683$$

Using flexibility tables

$$f_{11} = \frac{1}{3}Lac\frac{1}{3}Lac$$

$$= \frac{1}{3} \times 3.9 \times 1 \times 1 + \frac{1}{3} \times 2.0 \times 1 \times 1 = 1.97$$

$$\delta_{10} = \frac{5}{12} \times 2 \times (-1)(3) + \frac{1}{6} \times 3.90[-1(3 + 5.25)] = -7.86$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} = 4 \text{ kN}$$

Moment at C, M_c

$$M_c = m_0 + m_1x_1 = 3.0 + (-1)(4) = -1 \text{ kN m}$$

Reactions:

$$R_B = -4 \times \frac{2.5}{5} + \frac{1 \times 5^2}{2 \times 5} = 0.5 \text{ kN}$$

$$R_A = 1 \times 5 - 0.5 = 4.5 \text{ kN}$$

Moments at critical points

Span CB: at any distance $x_1 = 0.5$ m

$$M_{x1} = 0.5 \times 0.5 - \frac{(0.5)^2}{2} = 0.125 \text{ kN m}$$

In span AC, similarly at a horizontal distance $x_2 = 1.5$ m, $M_{x2} = 0.625$ kN m and at a distance of 0.90 m, the value of $M_{x2} = 0.65$ kN m which is the maximum value in this span.

Shear V
At A

$$V_A = R_A\cos\theta - H_A\sin\theta$$

$$= 4.5 \times 0.7683 - (-1) \times 0.6401$$

$$= 0.9 \text{ kN}$$

At C

$$V_C \text{ (left) or } V_{CA} = V_A - wL_1\cos\theta$$

$$= 0.9 - (1 \times 3.9 \times 0.7683)$$

$$= -1.4 \text{ kN}$$

$$V_C \text{ (right) or } V_{CB} = R_A - wL_1$$

$$= 4.5 - 1 \times 3$$

$$= 1.5 \text{ kN}$$

At B

$$V_B = V_{CB} - wL_2$$
$$= 1.5 - 1 \times 2$$
$$= -0.5 \text{ kN}$$

Axial Effects N
Owing to variation in geometry, the stairs can be subject to axial loads.

$$A + A, \; N_A = \text{axial force at } A = -R_A \sin\theta - H_A \cos\theta$$
$$= 4.5 \times 0.6401 - (-1) \times 0.7683$$
$$= 5.94 \text{ kN Compression}$$

$$A + C, \; N_C = N_{CA} \text{ (left)} = N_A + wL_1 \sin\theta$$
$$= -5.941 + (1 \times 3.90 \times 0.6401)$$
$$= -4.02 \text{ kN Compression}$$

$$N_C = N_{CB} \text{ (right)} = -H_B = -4 \text{ kN}$$

Case (ii), all dimensions are the same, but a load of 1 kN/m is acting on the landing CB. All diagrams for flexibility must now be modified since m_0 diagram is altered.

M_0 at C

Take moment about B when $x_2 = 2$ m:

$$R_A \times 5 = 1 \times 2 \times 1; \quad R_A = \frac{2}{5}$$
$$R_B = \frac{3}{5}$$
$$m_{0c} = \frac{3}{5} \times 2 = 1.2 \text{ kN m}$$

Similarly when $x_2 = 1$ m from B

$m_0 = 1.1$ kN m and $x_2 = 1.6$ m from B
$m_0 = 1.28$ kN m which is the maximum
m_1 diagram of Case (i) is still the same

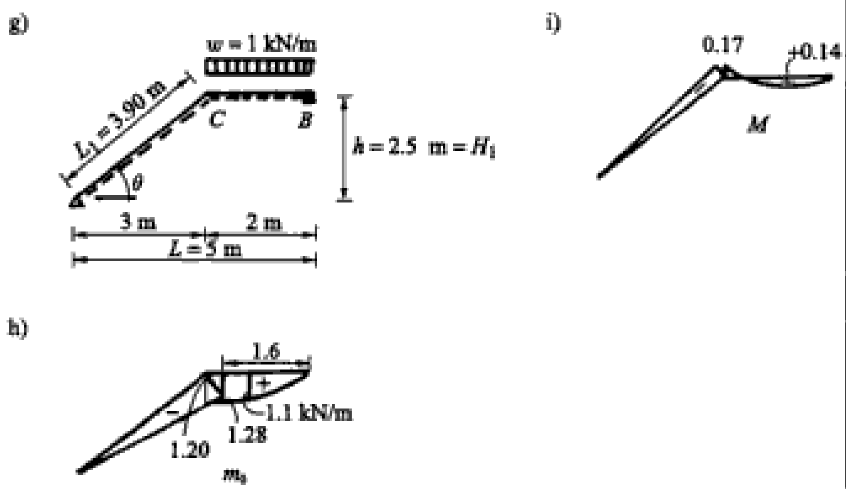


Figure 3.3. Flexibility diagrams for Case (ii).

$$\frac{\delta_{10}}{f_{11}} = X_1$$

$$\delta_{10} = \frac{1}{3} \times 3.90 \times 1.2(-1) + \frac{1}{6} \times 2[(-1 \times 1.2) + 2.2]$$

$$= -2.69$$

$$X_1 = 1.37 \text{ kN}$$

$$M = m_0 + m_1 X_1$$

$$M_c = 1.2 + (-1)(1.37) = 0.17 \text{ kN m}$$

For $X_2 = 1$ m

$$M_{x2} = 1.10 - 0.69$$

$$= 0.41 \text{ kN m}$$

Case (iii), when top and bottom supports are fixed and only the landing CB is loaded with 1 kN/m.

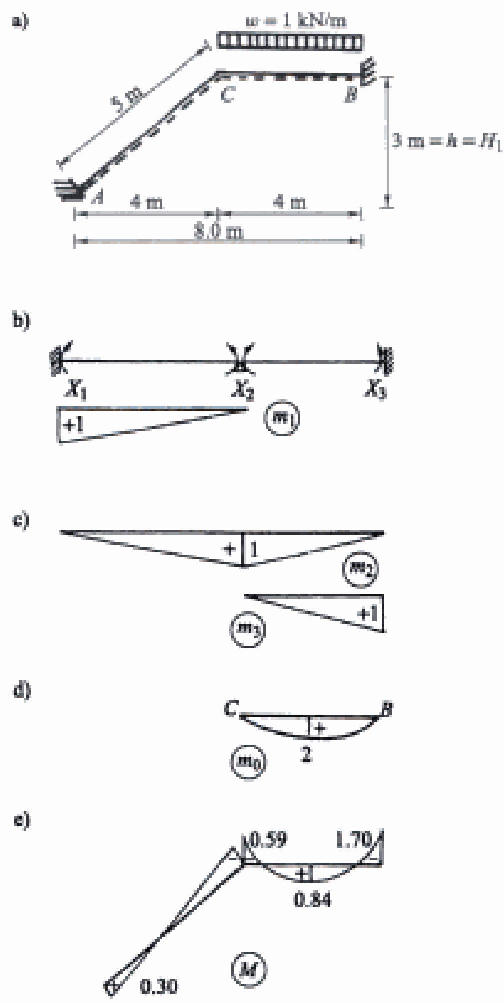


Figure 3.4. Flexibility diagrams for Case (iii).

$$f_{11} = \frac{1}{3}(5.0 \times 1 \times 1)$$
$$= \frac{5}{3}$$
$$f_{22} = \frac{1}{3}[(5 \times 1 \times 1) + (4 \times 1 \times 1)]$$
$$= 3$$
$$f_{33} = \frac{1}{3}(4 \times 1 \times 1)$$
$$= \frac{4}{3}$$
$$f_{13} = f_{31} = 0$$
$$f_{23} = f_{32} = \frac{1}{6}(4 \times 1 \times 1)$$
$$= \frac{2}{3}$$
$$f_{12} = f_{21} = \frac{1}{6}(5 \times 1 \times 1)$$
$$= \frac{5}{6}$$
$$\delta_{10} = 0$$
$$\delta_{20} = \delta_{30} = \frac{1}{3}(4 \times 1 \times 2)$$
$$= \frac{8}{3}$$

$$\begin{bmatrix} (f_{11}) & (f_{12}) & (f_{13}) \\ \frac{5}{3} & \frac{5}{6} & 0 \\ (f_{21}) & (f_{22}) & (f_{23}) \\ \frac{5}{6} & 3 & \frac{2}{3} \\ (f_{31}) & (f_{32}) & (f_{33}) \\ 0 & \frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} \delta_{10} = 0 \\ \delta_{20} = \frac{8}{3} \\ \delta_{30} = -\frac{8}{3} \end{Bmatrix}$$

$$X_1 = 0.30 \text{ kN m}$$
$$X_2 = -0.59 \text{ kN m}$$
$$X_3 = -70 \text{ kN m}$$
$$M = m_0 + m_1 X_1 + m_2 X_2 + m_3 X_3$$
$$M_B = 0 + (0 \times 0.30) + (0)(-0.59) - (1 \times 1.70)$$
$$= -1.7 \text{ kN m}$$
$$M_C = m_0 + m_1 X_1 + m_2 X_2 + m_3 X_3$$
$$= 0 + (0 \times 0.30) + (1 \times -0.59) + (0 \times -1.70)$$
$$= -0.59 \text{ kN m}$$
$$M_A = m_0 + m_1 X_1 + m_2 X_2 + m_3 X_3$$
$$= 0 + (1 \times 0.30) + (0 \times -0.59) + (0 \times -1.70)$$
$$= 0.30 \text{ kN m}$$

Figure 3.4(e) shows the final M diagram based on 1 kN/m. Assuming the dimensions are constant, any change in the load can simply be taken as new load kN/m. The respective values of M or others in the above 1 kN m cases can be enhanced by that factor.

Table 3.4. Deflection and rotation for wedge beam under concentrated loads.

1. Find the vertical deflection at B
   EI constant

$$\delta_B = \int_0^{2a} \frac{m_0 m_1}{EI}\,dx$$

$$= \frac{a}{6EI}[2(2Wa \times a) + Wa \times a]$$

$$= \frac{5Wa^3}{6EI}$$

2. Find the slope at B

$$\theta_B = \int_0^{2a} \frac{m_0 m_1}{EI}\,dx$$

$$= \frac{a}{6EI}[2(2Wa \times 1) + (2Wa \times 1) + 2Wa \times 1 + Wa \times 1]$$

$$= \frac{3Wa^2}{2EI}$$

3. Find the slope at the free end D

$$\theta_D = \int_0^{3a} \frac{m_0 m_1}{EI}\,dx$$

$$= \frac{a}{6(2EI)}[2(2Wa \times 1) + 2Wa \times 1 + 2Wa \times 1 + Wa \times 1] + \frac{a}{6EI}[2Wa \times 1 + Wa \times 1]$$

$$= \frac{5Wa^2}{4EI}$$

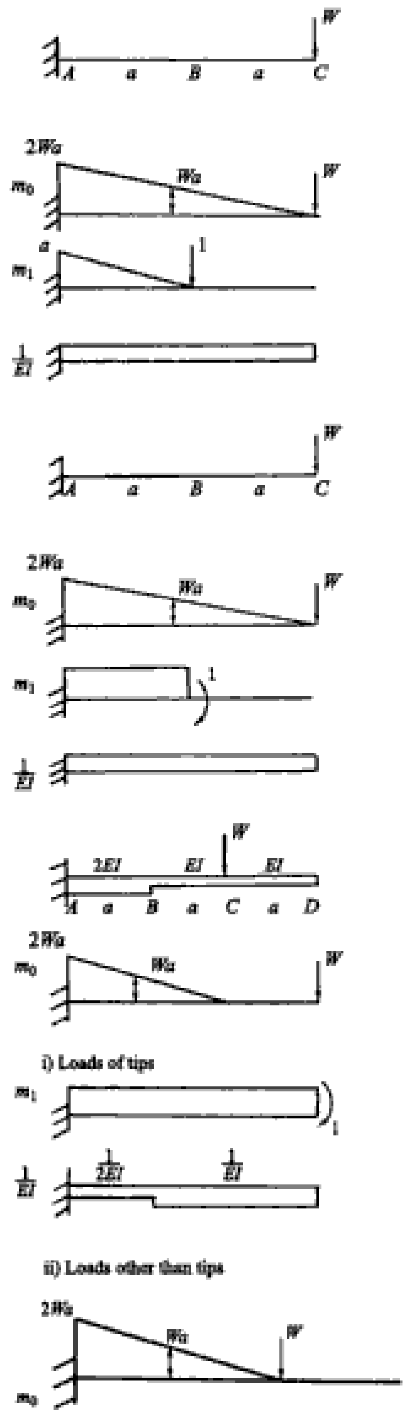


Figure 3.5. Stair girders or stringers under concentrated loads.

Table 3.5. A combined effect of three-stress resultant on a wedge beam.

A cantilever wedge beam is subject to an axial effect N, bending moment M and shear V as shown in Figures a)-e) for which the reference coordinates are given in Figure 3.6. It is assumed that EA and EI for this beam are constant.
For $N = 1$

$$f_{11} = \frac{L}{EA}, \quad f_{21} = 0, \quad f_{31} = 0$$

For $M = 1$

$$f_{12} = 0, \quad f_{22} = \frac{L}{EI}, \quad f_{32} = -\frac{L^2}{2EI}$$

For $V = 1$

$$f_{13} = 0, \quad f_{23} = -\frac{L^2}{2EI}, \quad f_{33} = \frac{L^3}{3EI}$$

$$[f] = \text{flexibility matrix} = \begin{bmatrix} (f_{11}) & (f_{12}) & (f_{13}) \\ \frac{L}{EI} & 0 & 0 \\ (f_{21}) & (f_{22}) & (f_{23}) \\ 0 & \frac{L}{EI} & -\frac{L^2}{2EI} \\ (f_{31}) & (f_{32}) & (f_{33}) \\ 0 & -\frac{L^2}{2EI} & \frac{L^3}{3EI} \end{bmatrix}$$

hence

$$\{D\} = \text{displacements} = \begin{Bmatrix} \delta_H \\ \theta \\ \delta_V \end{Bmatrix} = [f] \begin{Bmatrix} N \\ M \\ V \end{Bmatrix}$$

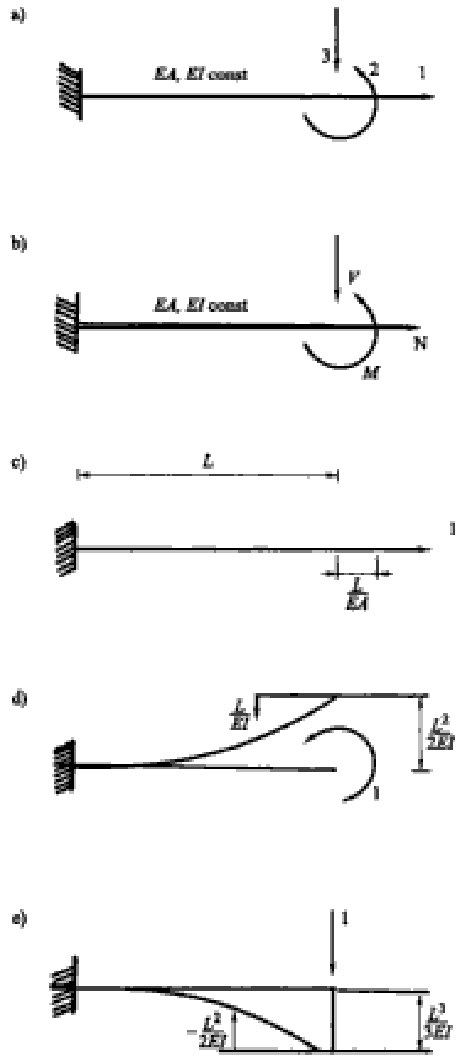


Figure 3.6. A wedge beam under three stress resultants.

> **EXAMPLE 3.2**
> The layout of a staircase with one flight and a landing is shown in Figure 3.7 in a horizontal plane.
> The structural layout of the building floor is such that a load W acts on the flight AC in its plane and which causes torsion along with bending. Using the flexibility method, calculate and draw the torsional moment. Assume EI and GJ constant throughout.

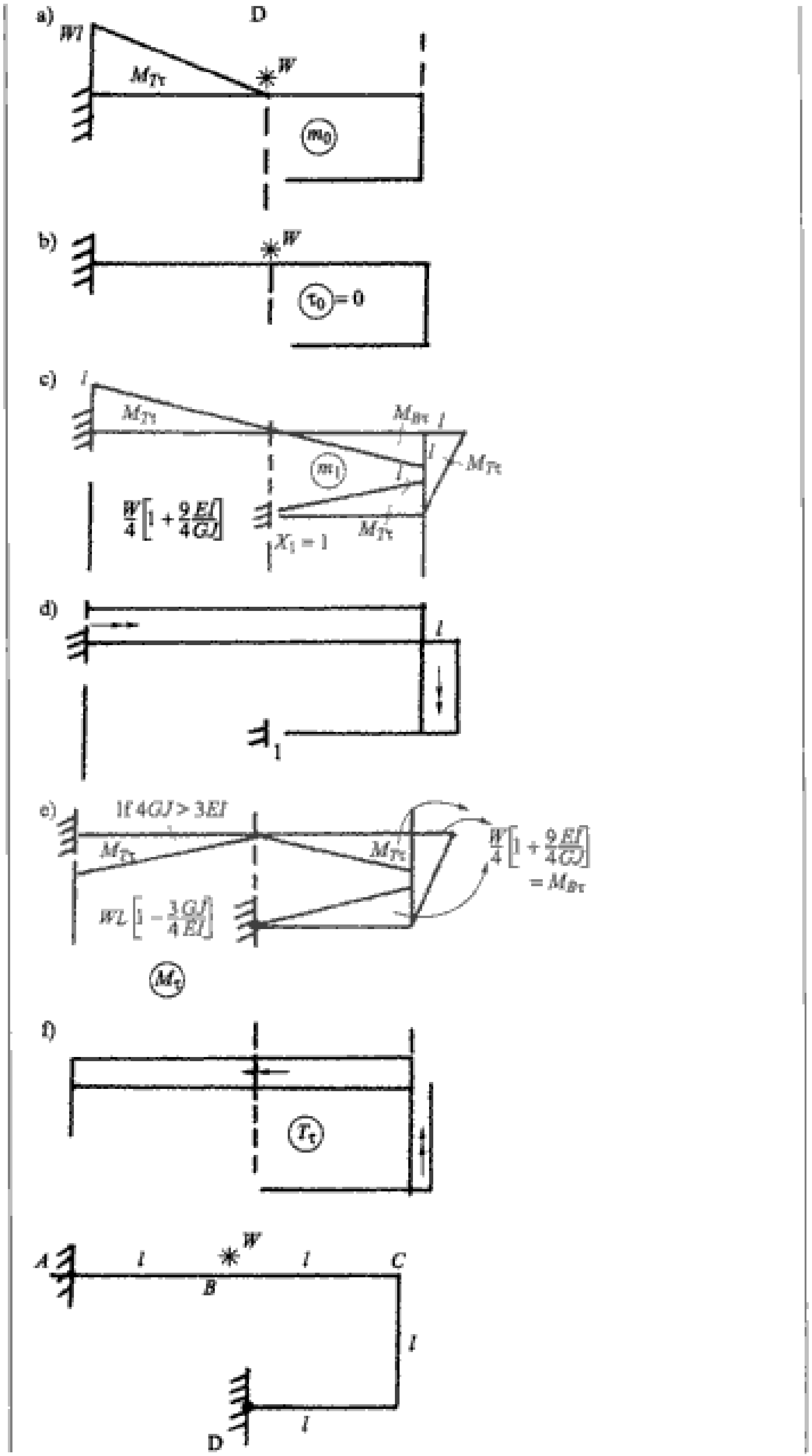


Figure 3.7. A horizontal plan view.

SOLUTION

One flight staircase under torsion.

At the ground floor, the flight is rigidly jointed and the landing is supported such that the end D is taken as pinned or simply supported.

Let M_B represent the moment at the bottom and M_t the moment at the top.

Flexibility diagrams are given in Figures 3.7(a) to (f)

$$f_{11} = \frac{4l}{6EI}\left[2(l)(l) + \frac{3l}{6GJ}(6l^2)\right]$$

$$= \frac{4l^3}{3EI} + \frac{3l^2}{GJ}$$

$$= \frac{4l^3}{3EI}\left[1 + \frac{9EI}{4GJ}\right]$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} = \frac{-w}{4}\left[1 + \frac{9EI}{4GJ}\right]$$

$$M = m_0 + m_1 x_1$$

Solutions are shown in Figures 3.7(e) and (f).

EXAMPLE 3.3

A stringer beam is supported at ground level A and at the first floor level B. Due to the other building requirements, it becomes necessary to support this beam at any intermediate point C. The plane projection of the system with loads are shown in Figure 3.8. Using the following case studies, calculate moment of the stringer beam at C:

a) A and B are simply supported and the column is placed at the centre of the stringer beam. The load w (2 kN/m) acts on the entire beam. The floor at the support B sinks by 1 cm. Take $EI = 12.5 \times 10^7$ kN cm^2.

b) The column support at C due to constructional problems has been moved closer to A such that the CB is not greater than 1.5 times span CA. The load is kept the same as in a).

c) As in b) but the uniform load w acts on AC.

SOLUTION

Stringer beam analysis

a)

$$EIf_{11} = \int_0^L m_1^2\, ds = \frac{1}{3}\frac{L^3}{4}$$

$$EI\delta_{10} = \int_0^L m_1 m_0\, ds = \frac{5}{12}\frac{wL^2}{8} \times \frac{L}{4} \times L$$

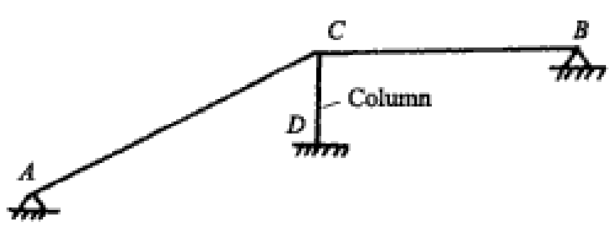


Figure 3.8. A single flight.

$$X_1 = -\frac{\delta_{10}}{f_{11}} = \frac{5}{8}wL$$

If any span $AC = L$, then

$$X_1 = \frac{5}{4}wL_1$$

$$M_C = m_0 + m_1 X_1$$
$$= \frac{wL^2}{8} + \frac{5}{4}wL_1\left(-\frac{L}{4}\right) = -\frac{wL_1^2}{8}$$

$w = 2$ kN/m $\quad L = 10$ m

$M_C = -25$ kN m

$$V_0 = \frac{wL}{2} = 20 \text{ kN}$$

$$X_1 = \frac{5}{4} \times 2 \times 10 = 25 \text{ kN}$$

$$V_C = \frac{1}{2} \text{ kN}$$

$$V = v_0 + v_1 x_1 = \frac{3}{8}wL_1 = 7.5 \text{ kN}$$

if the support sinks by 1 cm

$$X_1 = -\frac{\delta_1 s}{f_{11}} = -0.1875 \text{ kN}$$

$$M_C = m_0 + m_1 x_1 = 0 + 10(-0.1875) = -1.875 \text{ kN m}$$

$$\text{final } X_1 \doteq \frac{5}{8}wL_1 - 0.1875 = 24.8125 \text{ kN}$$

$$\text{final } M_c = -\frac{wL^2}{8} - 1.875 = -26.875 \text{ kN m}$$

Figures 3.8(g) to (k) give the step by step procedure

b)

$L_1 = 10$ m, $\quad L_2 = 15$ m

(Figs 3.8(l) to (p))

$$EIf_{11} = \frac{1}{3}Lac = \left(\frac{1}{3} \times 10 \times 1 \times 1\right) + \left(\frac{1}{3} \times 15 \times 1 \times 1\right) = \frac{25}{3}$$

$$EI\delta_{10} = \left(\frac{1}{3} \times 10 \times 25 \times 1\right) + \left[\frac{1}{3} \times 15 \times 1\left(2 \times \frac{15}{8}\right)^2\right] = \frac{1094}{3}$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} = -43.8 \text{ kN m} = M_c$$

c) A reference is made to Figures 3.8(q) to (y)

$$EIf_{11} = \left(\frac{1}{3} \times 10 \times 1 \times 1\right) + \left(\frac{1}{3} \times 15 \times 1 \times 1\right) = \frac{25}{3}$$

$$EI\delta_{11} = \frac{1}{3} \times 10 \times 25 \times 1 = \frac{250}{3}$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} = -10 \text{ kN m}$$

$$M = m_0 + m_1 X_1$$

$$M_C = 0 + 1 \times X_1$$
$$= X_1 = M_C = -10 \text{ kN m}$$

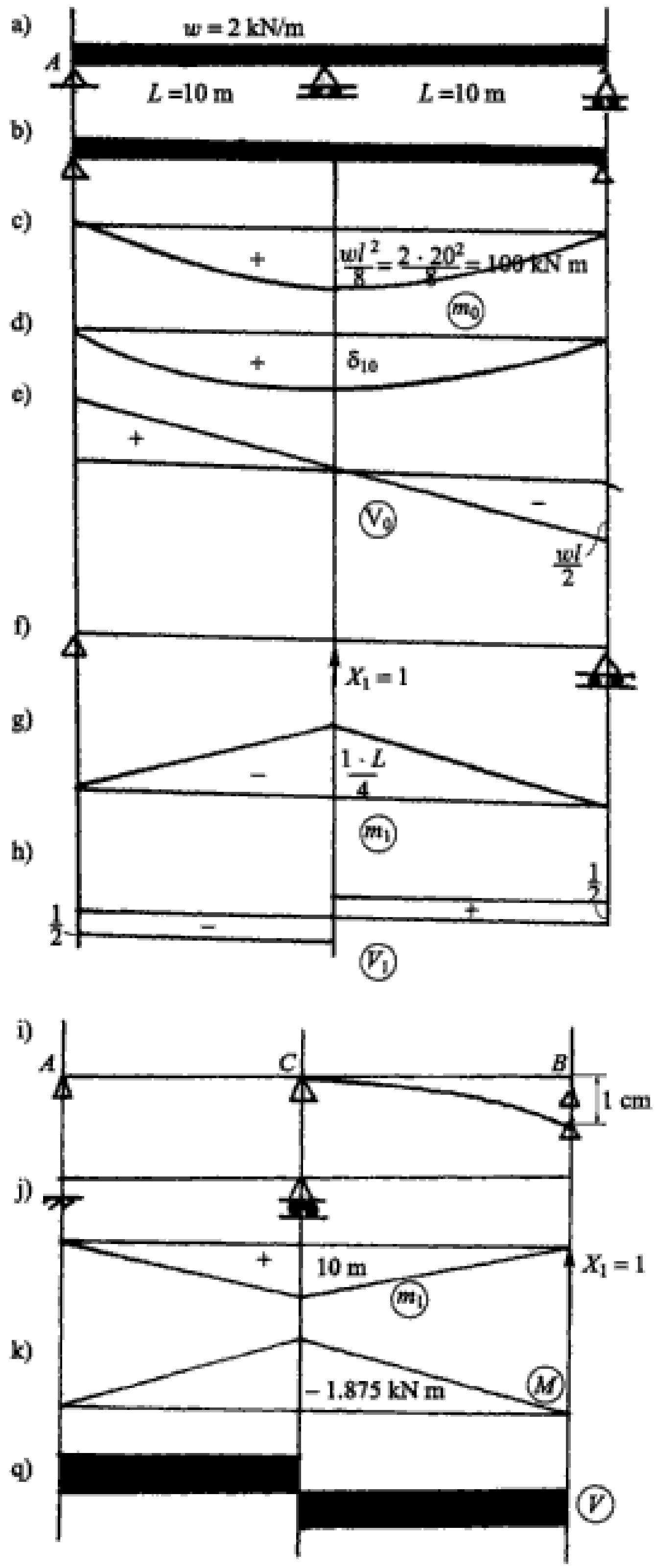


Figure 3.8 (cont.).

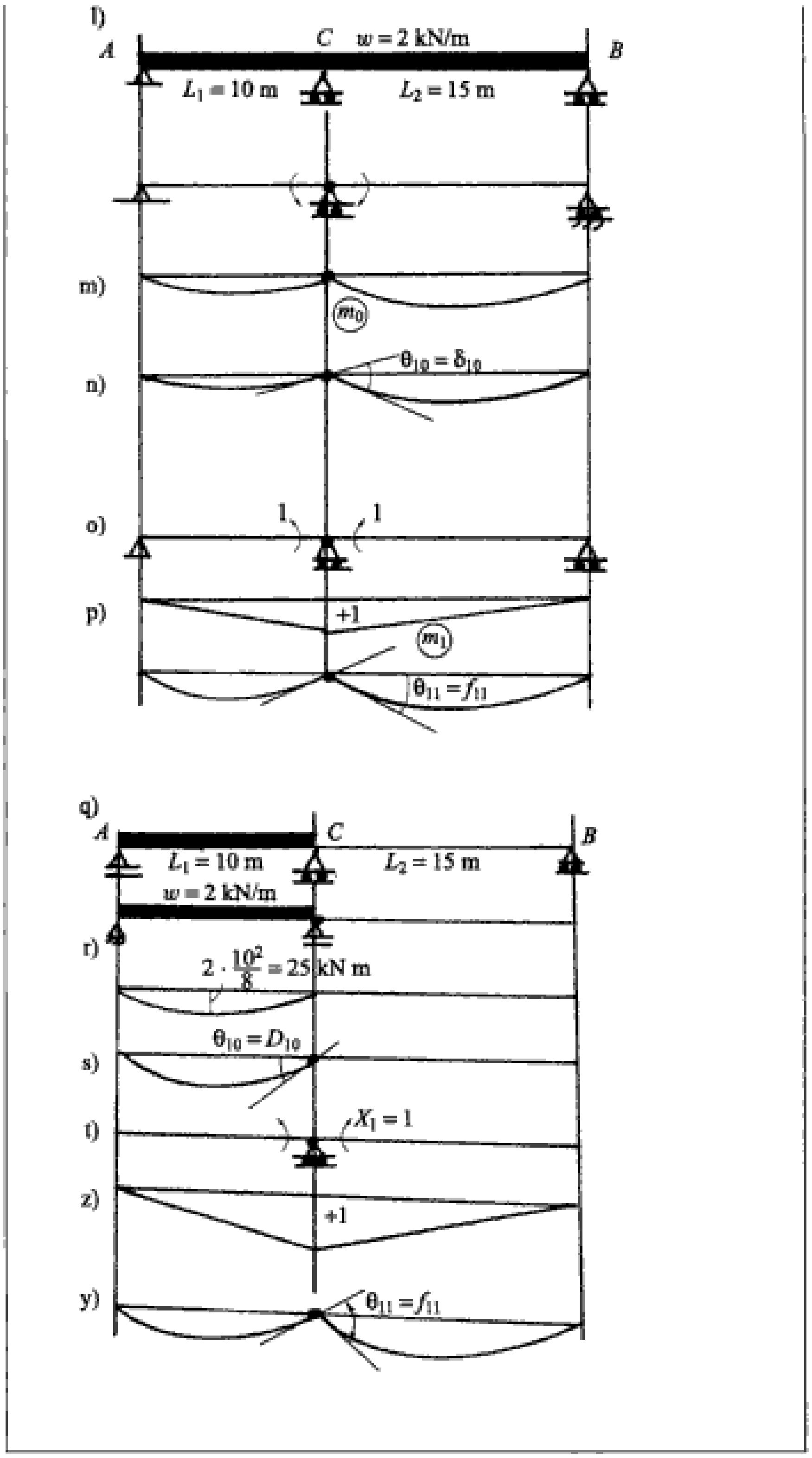


Figure 3.8 (cont.).

3.2.2 *Analysis of slabless treads-riser stairs (Saenz and Martin method 1961)*

In this analysis it is assumed that, owing to the possibility of the existence of a rigid beam at the beginning of the landings or a thick part of the slab at the ends, the stair is fixed at the end. For architectural reasons the treads are of the same size and are even or odd in number. Figure 3.9 shows that when the stairs are cut in the middle and $X_1 = 1$ at this cut or section, the M_1 diagram is constructed in the usual manner as described earlier. Generally bending moments, shear forces and axial forces are developed. Since the loads are symmetrical and the stairs in Figure 3.10 are unsymmetrical the values of V and N are zero, and hence only the M_0 diagram is constructed. This is shown in Figure 3.10

$$f_{11} = \int \frac{m_1^2}{EI}\,\mathrm{d}s, \quad \delta_{10} = \int \frac{m_1 m_0}{EI}\,\mathrm{d}s \tag{3.10}$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} \tag{3.11}$$

Odd and even number of treads:

odd number of treads $a = 2n + 1$ (3.12a)

even number of treads $a = 2n$ (3.12b)

for the even number of treads $P/2$ load is taken into account at the top of the middle riser.

Odd number of treads:

$$f_{11} = \frac{2L_1}{EI_L}(C_1 + \widehat{K}C_2)$$

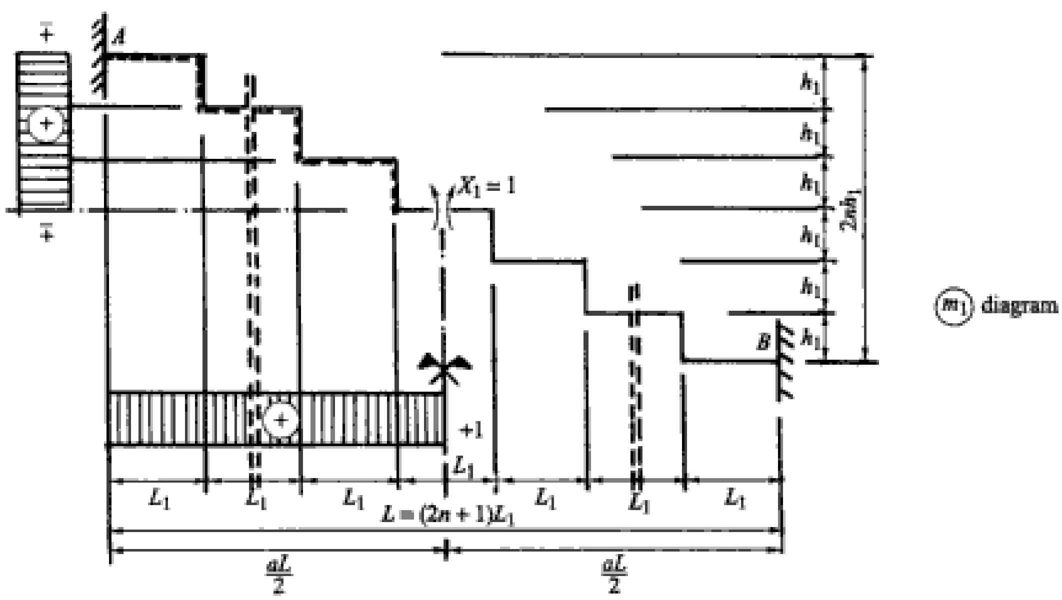


Figure 3.9. Tread-riser stairs.

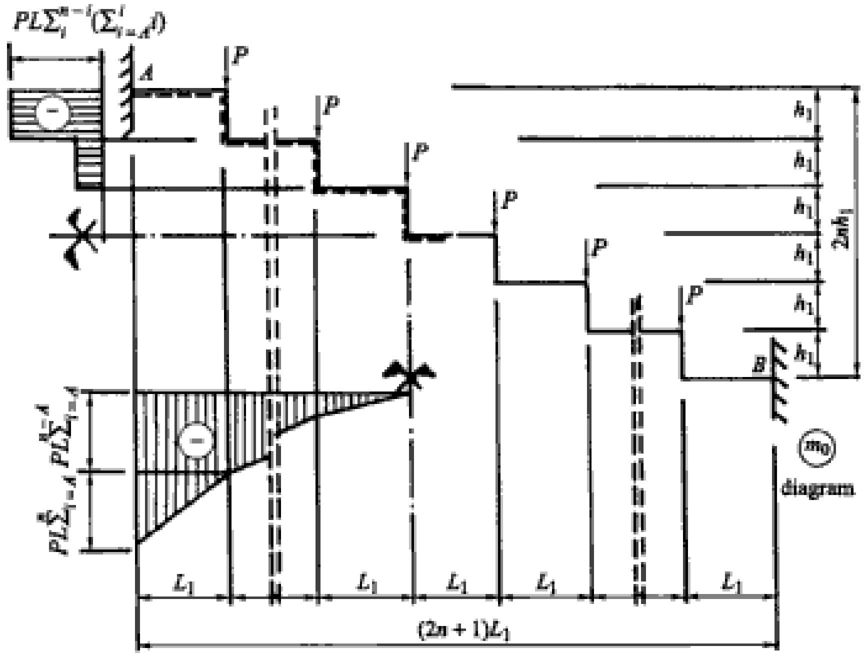


Figure 3.10. Moment diagram for external loads P.

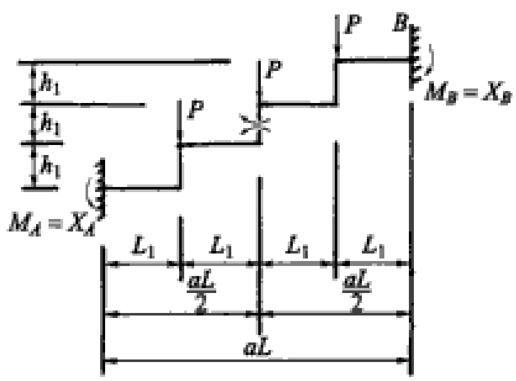


Figure 3.11. Even number of treads.

Where

$$C_1 = \frac{2n+1}{2}, \quad C_2 = n, \quad \widehat{K} = \frac{h_1}{L_1} \times \frac{I_{L1}}{I_{h1}}, \quad K = 1 + \hat{k} \tag{3.13}$$

$$-\delta_{10} = \frac{2PL_1^2}{EI_{L1}}\left(C_3 + \widehat{K}C_4\right) \tag{3.14}$$

Tables 3.6 and 3.7 give the values of C_1 to C_4 where

$$C_3 = \frac{n(n+1)}{4}, \quad C_4 = \frac{n(n^2-1)}{6} \tag{3.15}$$

Table 3.6. Odd treaded stairs.

Coefficients	No. of treads $ds = a$								
	3	5	7	9	11	13	15	17	19
C_1	1.50	2.50	3.50	4.50	5.50	6.50	7.50	8.50	9.50
C_2	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
C_3	0.50	1.50	3.00	5.00	7.50	10.50	14.00	18.00	22.50
C_4	0.00	1.00	4.00	10.00	20.00	35.00	56.00	84.00	120.00

Table 3.7. Even treaded stairs.

Coefficients	No. of treads $ds = a$								
	2	4	6	8	10	12	14	16	18
C_1	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
C_2	0.50	1.50	2.50	3.50	4.50	5.50	6.50	7.50	8.50
C_3	0.25	1.00	2.25	4.00	6.25	9.00	12.25	16.00	20.25
C_4	0.00	0.50	2.50	7.00	15.00	27.50	45.50	70.00	102.00

Hence

$$X_1 = M \tag{3.16}$$

$$X_1 = M = -\frac{\delta_{10}}{f_{11}} = PL_1 \frac{C_3 + \widehat{K}C_4}{C_1 + \bar{k}C_2} \tag{3.17}$$

mid span moment when it is simply supported

$$M = m_0 = 2\left\{\frac{n(n+1)}{4} PL_1\right\}$$

so

$$M_A = M_B = X_A = X_B = 2C_3PL_1 - m_0 \tag{3.18}$$

Even number of treads:
Now take $P/2$ load at the top of the riser (at the top of the middle riser), the coefficients C_1, C_2, C_3 and C_4 are changed for even numbers. In a similar manner each one of them is evaluated. These values are given below:

$$\begin{aligned} &C_1 = n, \quad C_2 = \frac{2n-1}{2}, \quad C_3 = \frac{n^2}{4}, \\ &C_4 = \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{4} \end{aligned} \tag{3.19}$$

On the basis of the above equations, Tables 3.6 and 3.7 are prepared for odd and even treads for stairs.

EXAMPLE 3.4
Calculate moment at the fixed ends for a staircase. Using the following data:

$$P = 2.58 \text{ kN}$$

$$a = 12 \text{ even number treads}$$

$$L_1 = 0.279 \text{ m}$$

$$I_{L1} = 853 \times 10^{-7} \text{ m}^4$$

$$I_{h1} = 1667 \times 10^{-7} \text{ m}^4$$

$$h_1 = 0.178$$

SOLUTION
Slabless stairs
A reference is made to Table 3.7

for $a = 12$ coefficients $C_1 = 60$, $C_2 = 5.5$, $C_3 = 9.0$, $C_4 = 27.50$

$$\begin{aligned} m_0 &= PL_1 \frac{C_3 + (1+\hat{k})C_4}{C_1 + C_2\hat{k}} \\ &= 2.58 \times 0.279 \frac{9.0 + (1 + 0.3265) \times 27.50}{6.0 + 5.5 \times 0.3265} \\ &= 0.71982\left(\frac{45.47875}{7.79575}\right) = 4.2 \text{ kN m} \end{aligned}$$

$$X_A = M_A$$

$$\begin{aligned} X_B = M_B &= 2C_3PL_1 - m_0 \\ &= (2 \times 9 \times 2.58 \times 0.279) - 4.2 \approx 8.757 \text{ kN m} \end{aligned}$$

Boundary conditions
I. When the landing exists on both sides in a staircase
Figure 3.12 shows a staircase in which A and B form the centres of two opposing landings such that symmetrical rotation can occur at these points, which act as the end supports of the staircase. When $\theta_A = \theta_B = 1$ at these points, the resulting rotation at the mid span section will be $\delta_{1w} = 2$.

$$M_0 = \text{mid span moment} = X = \frac{-\delta_{1w}}{f_{11}} = -\frac{2}{f_{11}} \tag{3.20}$$

$$= \frac{-2}{2\frac{L_1}{EI_{L1}}(C_1 + \hat{k}C_2)}$$

$$= \frac{-EI_{L1}}{L_1(C_1 + \hat{k}C_2)} \tag{3.21}$$

$$= K_{AB} = K_{BA}$$

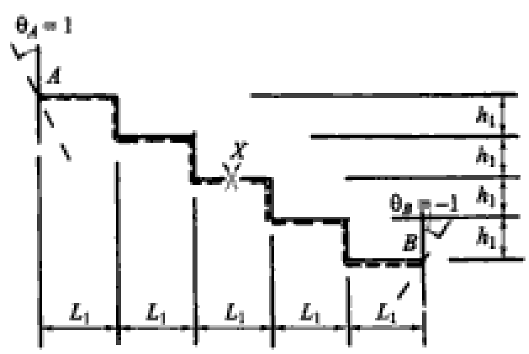


Figure 3.12. Rotations allowed at the far end of a staircase.

The rest of the procedure is the same as given earlier.

II. When the far ends carrying landings of a staircase are fixed and additional supports exist at the other ends of the landings

Figure 3.13 shows a typical staircase where landings are loaded with uniform loadings.

The stiffness is given by Eq. (3.21) for K_{AB} or K_{BA} (3.22)

for the landing AA_1, the stiffnes $K_{AA_1} = \dfrac{4I_{LY}}{L_s}$

Where I_{LV} = second moment of area of the landing at vertical section.

L_s = landing span

The distribution factor DF_{A_1A} from Equations (3.21) and (3.22) is given by

$$DF_{A_1A} = \frac{4}{4 + \dfrac{K^*}{C_1 + kC_2}}$$

for the landing only where

$$K^* = \frac{I_{L1}}{I_{h1}\left(\dfrac{L_s}{L_1}\right)} \tag{3.23}$$

The distribution factor of slabless $= DF_{AB} = 1 - DF_{A_1A}$ (3.24)

tread riser staircase AB.

Owing to a symmetrical deformation, the moment distribution at support A is required.

$$M_{AB} = X_{AB} = -X_{AA} = DF_{AA_1}M^F_{AB} - DF_{AB}M^F_{AA_1} \tag{3.25}$$

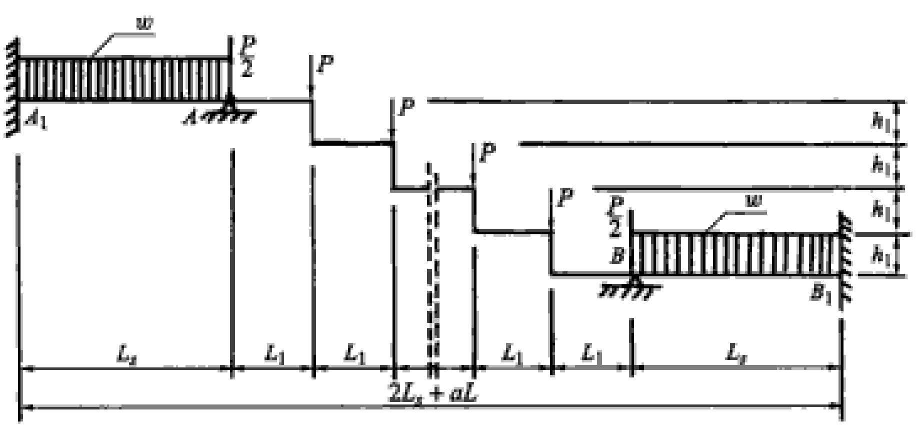
M_{A_1A} = the final moment at landing A_1A

$$= M^F_{AB} - DF_{AA_1}\left\{\frac{M^F_{AB} + M^F_{AA_1}}{2}\right\} \tag{3.26}$$

mid span moment $M = m_0 + \left(M^F_{AB} + M^F_{AA_1}\right)DF_{AB}$ (3.27)

reactions $R_A = C_1P + \dfrac{wL_s}{2} - \left\{\dfrac{M_{AA_1} + M_{A_1A}}{L_s}\right\}$ (3.28)

Figure 3.13. A staircase with concentrated loads P on steps and uniform loads on landings.



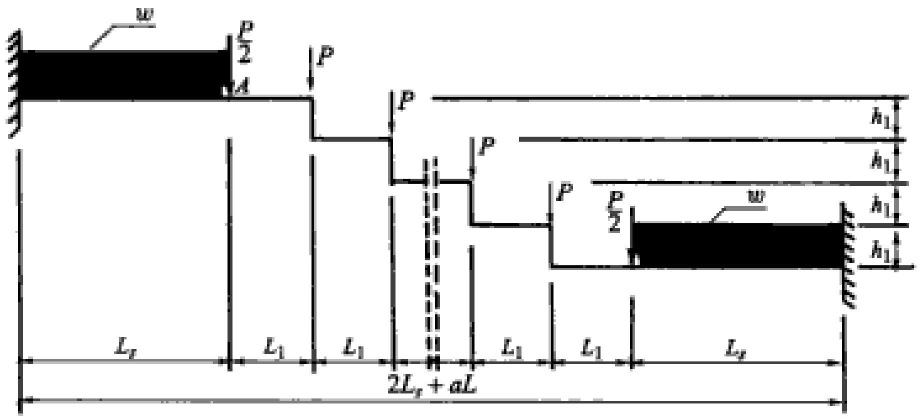


Figure 3.14. Stairs with equal landings with no supports at A and B.

Figure 3.14 shows the staircase where the supports at A and B are removed. The load is added at A and B and it is assumed they have equal deformation. The vertical deflections at A and B are equal and the moments reactions are:

$$M^F_{A_1A} = 1 - \frac{1}{2}DF_{A_1A}$$

$$M_{AB} = -DF_{AB}$$

$$R_A = \text{reactions due to vertical placement}$$

$$= \frac{1 + 3DF_{AB}}{2L_s} \tag{3.29}$$

When supports are removed, the value of $R_A = 0$, hence the final moments become:

$$M_{A_1A} = M^F_{A_1A} - \frac{DF_{A_1A}}{2}(M^F_{AB} - M^F_{AA_1}) = \frac{R_A}{R'_A}\left(1 - \frac{1}{2}DF_{A_1A}\right) \tag{3.30}$$

$$M_{AB} = DF_{A_1A}M^F_{AB} - DF_{AB}M^F_{A_1A} - \frac{R_A}{R'_A}DF_{AB} \tag{3.31}$$

$$M = m_0 + (M^F_{AB} + M^F_{A_1A})DF_{A_1A} + \frac{R_A}{R'_A}DF_{AB} \tag{3.32}$$

EXAMPLE 3.5

Calculate final moments for the staircase using case studies I and II and the following data:

$$P = 2.58 \text{ kN}$$

$$a = 12$$

$$L_1 = 0.279 \text{ m}$$

$$I_L = 853 \times 10^{-7} \text{ m}^4$$

$$L_s = 2.0 \text{ m}$$

$$w = 7.35 \text{ kN/m}$$

$$h_1 = 0.178 \text{ m}$$

$$I_{h1} = 1667 \times 10^{-7} \text{ m}^4$$

$$I_L = I_{LS}$$

Assume clockwise moments are positive.

SOLUTION

Staircases with specific boundary conditions.

$$\overline{K} = \frac{I_{L1}h_1}{I_{h1}L_1} = 0.3265$$

for $a = 12$ coefficients $C_1 = 6.0$, $C_2 = 5.5$, $C_3 = 9.0$, $C_4 = 27.5$

$$m_0 = PL_1 \frac{C_3 + (1+\overline{K})C_4}{C_1 + \overline{K}C_2} = 5.834 \text{ kN m}$$

$$\overline{K} = \frac{I_{L1}L_s}{I_{h1}L_1} = 7.1685$$

$$DF_{A_1A} = \frac{4}{\left(4 + \dfrac{7.1685}{6 + 5.5 \times 7.1685}\right)} = 0.962$$

$$DF_{AB} = 1 - DF_{A_1A} = 0.038$$

$$X_A = M_{AB} = 7.123 \text{ kN m}$$

$$X_{AA_1} = M^F_{AA_1} = -\frac{wL_s^2}{12} = \frac{-7.35 \times 2.0}{12} = -1.225 \text{ kN m}$$

$$X_{A_1A} = M^F_{A_1A} = 1.225$$

$$M_{AB} = -0.962 \times 7.123 = 0.038(-1.225) = 6.90 \text{ kN m}$$

$$M_{A_1A} = 1.225 - [7.123 + (-1.225)] \times \frac{0.962}{2} = -1.504 \text{ kN m}$$

$$m_{01} = M = 5.834 + (7.123 - 1.225) \times 0.038 = 6.058 \text{ kN m}$$

$$R_A = 2 \times 2.58 + 7.35 \times \frac{2}{2} - (-1.504 - 7.123)/2.0 = 33.61 \text{ kN}$$

3.3 SLABLESS STAIRCASE ANALYSIS UNDER UNIFORM LOAD

The slabless tread-riser stairs with the far ends fixed are subject to a uniform load $w = w_D + w_L$ where w_D is a factored dead load and w_L is a factored imposed load. The flexibility method is again adopted. Figure 3.15 shows the flexibility diagrams on the lines suggested in the preamble.

Both ends are restrained and

L = total horizontal length

$$= (n+1)L_1 \tag{3.33}$$

moment of inertia at any reference point $= I_0$

$$I_0 = \frac{I_{h1}}{I} \tag{3.34}$$

hence the I_0/I for the riser $= 1$

by symmetry $X_1 = X_2$ (3.35)

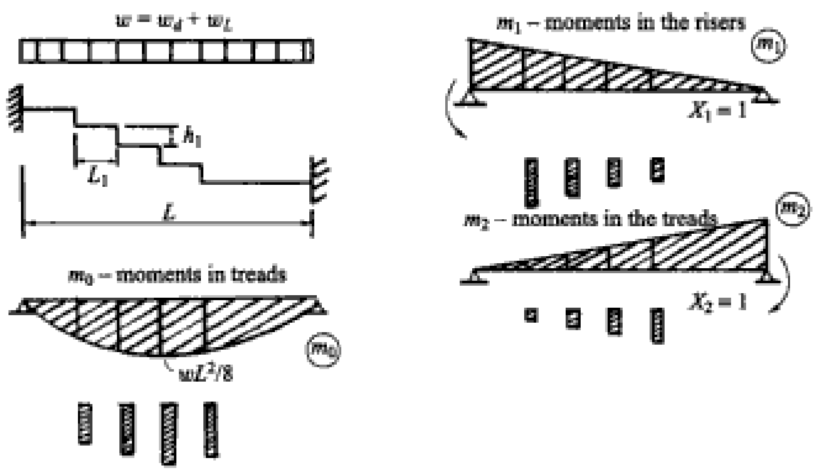


Figure 3.15. Flexibility diagram for a uniformly loaded slabless stair.

$$m_0 = \frac{wL^2}{8} EI \int m_1 m_0 \, ds \tag{3.36}$$

$$EI_h f_{11} = -\frac{1}{3} m_0 m_1 L \left(\frac{I_{h1}}{I_{L1}} + \frac{h_1}{L_1} \right) \tag{3.36a}$$

$$EI_{h1} f_{11} = \frac{1}{2} m_1^2 L \left(\frac{I_{h1}}{I_{L1}} + \frac{h_1}{L_1} \right) \tag{3.36b}$$

$$X_1 = -\frac{\delta_{10}}{f_{11}} = \frac{-\frac{1}{3} m_0 m_1 L \left(\frac{I_{h1}}{I_{L1}} + \frac{h_1}{L_1} \right)}{\frac{1}{2} m_1^2 L \left(\frac{I_{h1}}{I_{L1}} + \frac{h_1}{L_1} \right)} \tag{3.37}$$

since $m_1 = 1$

$$X_1 = \frac{2}{3} m_0 = \frac{2}{3} \frac{wL^2}{8} = \frac{wL^2}{12} \tag{3.38}$$

It is interesting to note that the fixed end moment for the symmetric slabless stairs, without landings, is equal to the fixed end moment of a straight clamped beam with the same span and load. The simplification of distributing the height of the riser between the length of the tread is acceptable with a high degree of accuracy provided the stairs have more than 4 treads. Hence the uniform load can be written into a concentrated load as:

$$w = \frac{P}{L_1} \tag{3.39}$$

EXAMPLE 3.6

Examine, in the light of the above technique, a slabless stairs using the following data:

$$P = 2.58 \text{ kN}$$
$$a = 12$$
$$L_1 = 0.279 \text{ m}$$
$$I_{L1} = 853 \times 10^7 \text{ m}^4$$
$$h_1 = 0.178 \text{ m}$$
$$I_{h1} = 1667 \times 10^{-7} \text{ m}^4$$

SOLUTION

Slabless stairs

From previous calculations

coefficients $C_1 = 6$, $C_2 = 5.5$, $C_3 = 9.0$, $C_4 = 27.50$, $\hat{k} = 0.3265$

$$m_0 = PL_1 \frac{C_3 + (1 + \hat{k})C_4}{C_1 + C_2\hat{k}} = 4.2 \text{ kN m}$$

$$w = \frac{P}{L_1} = 9.2473 \text{ kN m}$$
$$L = 12 \times 0.279 = 3.348 \text{ m}$$
$$m_0 = 9.2473 \times \frac{3.348^2}{8} = 12.956743 \text{ kN m} \sim 12.96 \text{ kN m}$$
$$X_1 = M_{AB} = \frac{2}{3}m_0 = 8.64 \text{ kN m}$$

The value of $X_1 = X_A = X_B = 8.754$ kN m from the previous method. They both are in agreements the error being 1.2%.

3.3.1 *A generalised case of even and odd number of treads with variable thickness in slabless stairs*

Consider a slabless stairs with any number of steps or treads 'n', where 'n' may be odd or even. Let the thickness of the tread be t_0 and that of the riser be t_r. Again, the value of $\overline{K}$ will be written as:

$$\overline{K} = \left(\frac{I_{L1}}{I_{h1}} + \frac{h_1}{L_1}\right)$$
$$\text{hence } I_{L1} = \frac{L_1 t_h^3}{12} \quad \text{and} \quad I_{h1} = \frac{t_r h_1^3}{12} \tag{3.40}$$

The fixed end moment M at any end can be represented (Fig. 3.16):

$$M = \frac{PnL_1(n^2 - 1)}{12n} \times \left[\frac{1 + \overline{K}}{1 + \frac{n-1}{n}\overline{K}}\right] \tag{3.41}$$
$$= M^F K_B$$

Equation (3.41) contains two terms: the fixed end moment M^F for a simple structure which is outside the brackets and K with a ratio of n in

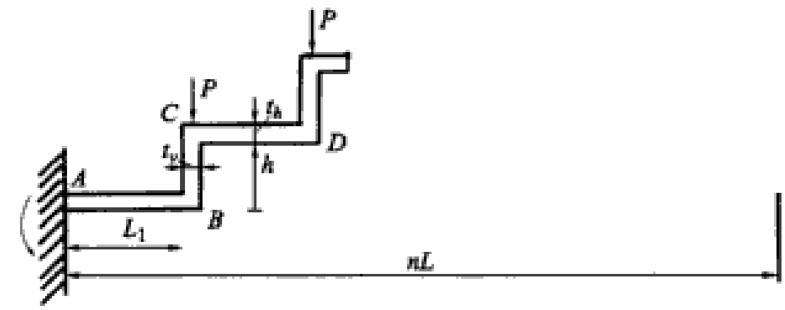


Figure 3.16. Slabless stairs with variable thicknesses of riser and treads.

the bracket. The variation of 'K_B' versus 'n' can easily be determined and plotted. Figure 3.17 shows a plot for K_B versus n for various ratios of h_1/L_1.

EXAMPLE 3.7

Solve the staircase given in the example above using the above equation with constant and variable thicknesses.

The following data can be used for the solution of this problem:

$P = 2.58$ kN, $n = a = 12$

Data 1

$L_1 = 0.279$ m, $\frac{t_h^3}{t_r} = 0.01045$

$\frac{h_1}{L_1} = 0.638$, $\frac{I_{h1}}{I_{L1}} = 0.5117$

Data 2

$\frac{t_r}{t_h} = 1$, $\frac{I_{h1}}{I_{L1}} = 1$

$\frac{h_1}{L_1} = 0.638$

the section of the riser and the tread has to resist exactly the same bending moment.

SOLUTION

Data 1:

$\overline{K} = 0.3265$

$$K_B = \frac{1 + 0.3265}{1 + \frac{11}{12} \times 0.3265} = 1.0209$$

$$M = M^F K_B$$

$$= \frac{2.58 \times 12 \times 0.279(12^2 - 1)}{12 \times 12} \times 1.0209 = 8.757 \text{ kN m}$$

graphically computed

$K_B = 1.021$

$M = 8.757$ kN m

Data 2:

$$\overline{K} = \frac{h_1 I_{L1}}{L_1 I_{h1}} = 1$$

$$K_B = \frac{1 + 1}{1 + \frac{11}{12} \times 1} = 1.04348$$

$$M = M^F K_B$$

$$= \frac{2.58 \times 12 \times 0.279(12^2 - 1)}{12 \times 12} \times 1.04348 = 8.951 \text{ kN m}$$

graphically computed

$K_B = 1.044$

$M = 8.957$ kN m

The two results from Data 1 and 2 show very little difference when the thicknesses are different for the same ratio of h_1/L_1. For technical reasons, both may be acceptable.

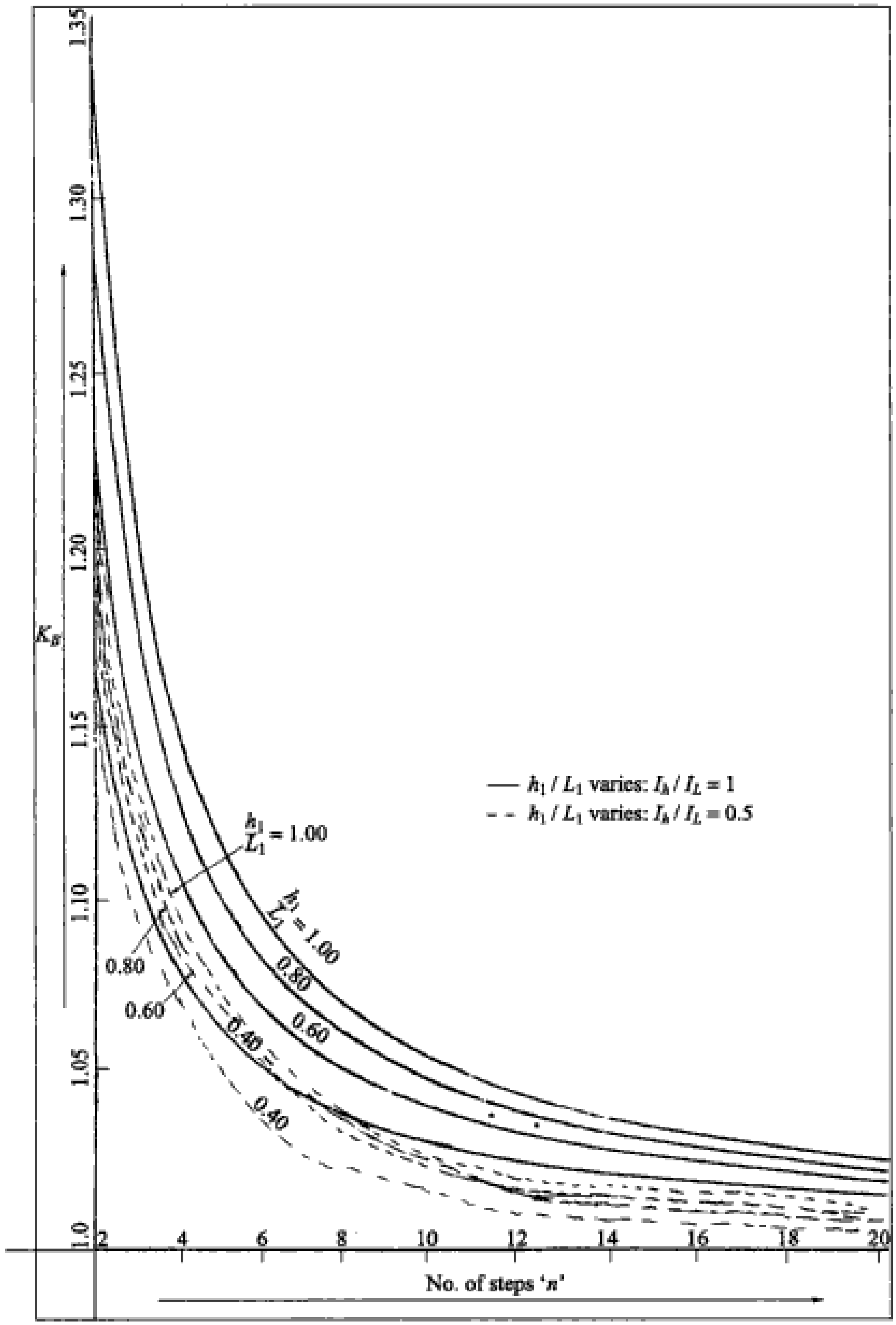


Figure 3.17. Plotted values of the variation of K with n.

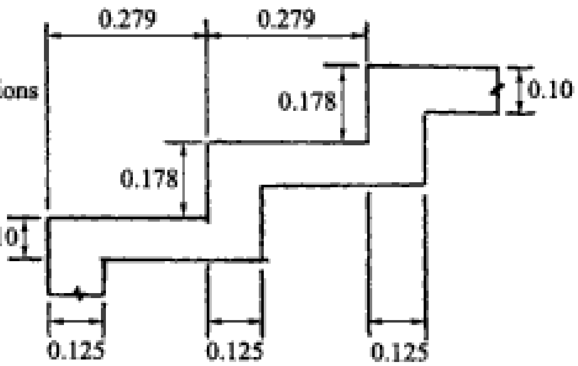


Figure 3.18. A slabless staircase.

EXAMPLE 3.8

Figure 3.18 shows treads and risers for a staircase. Using the given dimensions, calculate I_{t1} and I_{h1} for each step and determine their ratio. Use the stairway width as 1.0 m.

SOLUTION

$$I_{L1} = \frac{1.0 \times 0.10^3}{12} = 833 \times 10^{-7}\ \text{m}^4$$

$$I_{h1} = \frac{1.0 \times 0.125^3}{12} = 1628 \times 10^{-7}\ \text{m}^4$$

$$\frac{I_{h1}}{I_{L1}} = 1.954$$

3.3.2 *Free standing staircases with different loadings – analysis*

Three different types of staircase with various boundary and loading conditions are shown in Figure 3.19. In Figure 3.19(a) only top and bottom landings are simply supported at far ends with live loads q_p and q_L. Figure 3.19(b) shows a continuous support of the flight with cantilever landings.

q_L and q_p have relations. In some cases $q_p = 1/3q_L$ and $q_L = 1/3q_p$. Slight changes in notations were necessary so that each one in a specific span might be relatively identified.

Figure 3.19(c) shows all points of the staircase that are supported. The loads q_k and g_k represent imposed and dead loads. Figures 3.19(d) and (e) show generalised dimensions. The total load

$$w = \gamma_g g_k + \gamma_L q_k \tag{3.42}$$

where, γ_g = load factor for dead load
γ_L = load factor for imposed load.

The general bending moment is $wL^2/12$.

The general flexibility equation is written as:

$$f_{ij} = \int_s \frac{m_i m_j}{EI}\,\mathrm{d}s + \left\{ \int_s \frac{v_i v_j}{K_w} + \int_s \frac{T_i T_j}{K\theta} \right\} \mathrm{d}s \tag{3.43}$$

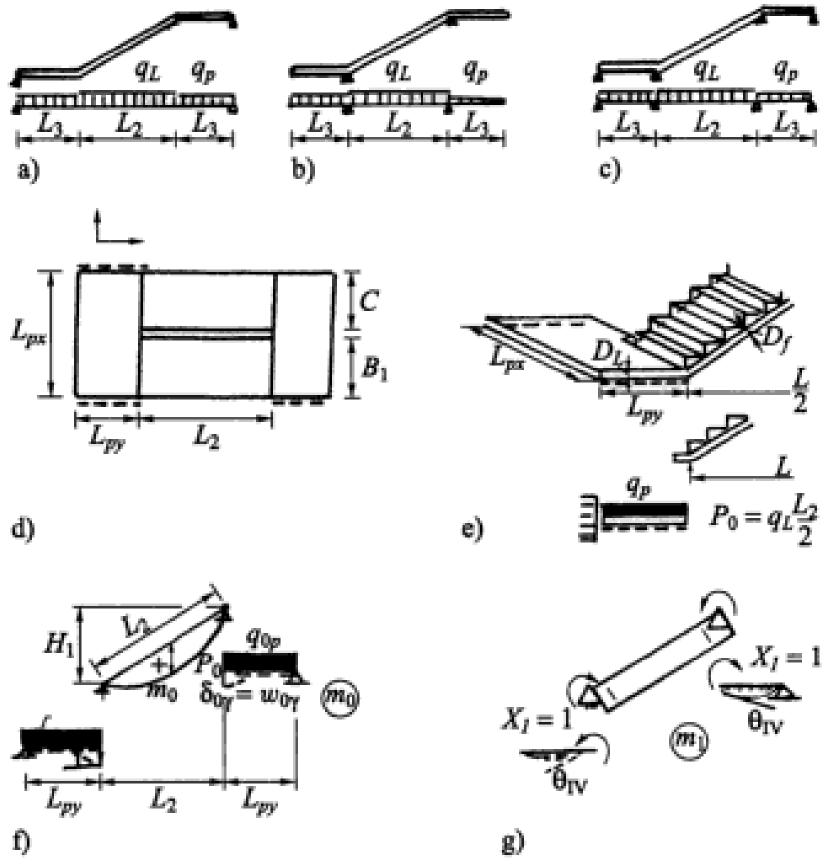


Figure 3.19. Stairs with different loadings.

$$\delta_{i0} = \int_s \frac{m_i m_0}{EI}\,\mathrm{d}s + \left\{ \int_s \frac{v_i v_0}{K_w} + \int_s \frac{T_i T_0}{K\theta} \right\} \mathrm{d}s \tag{3.44}$$

or

$$= \sum^n m_i m_0 + \sum^n v_i v_0^n + \sum^n T_i^n \theta_0^n \text{ with multiplying factor} \tag{3.45}$$

Looking at the landings and the flight in Figures 3.19(f) and (g) when $X_1 = 0$ or $X_1 = 1$, various displacements, rotations and moments are shown.

$$EIf_{11} = \frac{1}{2}L_2 + \beta\frac{L_{py}}{\bar{\theta}_{\mathrm{III}}}$$

$$EI\delta_{10} = \frac{q_L L_3^3}{24} - \beta\left(q_p\frac{L_{py}}{\bar{\theta}_{\mathrm{I}}} + p_0\frac{L_{py}^2}{\bar{\theta}_{\mathrm{II}}}\right) \tag{3.46}$$

where

$$\beta = \left(\frac{D_f}{D_L}\right)^3 \tag{3.47}$$

D_f = depth of the flight slab
D_L = depth of the landing slab

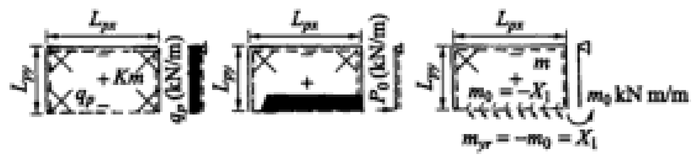
If both slabs have the same depth the value of $\beta = 1$. Tables 3.8 and 3.9 show various parameters for loading cases and boundary conditions.

$$X_1 = -\frac{\delta_{10}}{f_{11}} = -\frac{\dfrac{1}{24} - \beta\left[\left(\dfrac{q_p}{q_L}\right)\dfrac{1}{\bar{\theta}_{\mathrm{I}}}\left(\dfrac{L_{py}}{L_2}\right) + \dfrac{1}{2\bar{\theta}_{\mathrm{II}}}\left(\dfrac{L_{py}}{L_2}\right)^2\right]}{\dfrac{1}{2} + \dfrac{\beta}{\bar{\theta}_{\mathrm{III}}}\left(\dfrac{L_{py}}{L_2}\right)} \tag{3.48}$$

It should be remembered that q_L or q_p will have a value of w, the total load/unit length.

Table 3.8. Moments for two cases.

M_{X1} = right moment of $x = m_{Xi} + m_{Xii} + m_{Xiii}$
$m_i = m_{i\mathrm{II}} + m_{i\mathrm{III}}$
For Case I $= m_{xym} = m_{xm} = q_p L_{xp}^2/8$
$m_{ym} = 0$
$m_{yr} = -m_0 = X_1$



Case I   Case II   Case III

		0.3	0.4	0.5	0.6	0.07	0.8	0.9	1.0
II	m	2.19	2.75	3.17	3.45	3.65	3.81	3.88	9.6
	m	2.39	3.23	4.05	4.88	5.81	6.81	7.41	0.0
	m	38.5	31.3	27.8	26.4	25.7	26.4	27.1	9.8
	m	2.63	3.79	5.18	6.85	9.00	12.1	15.6	20.9
	m	10.0	9.6	9.2	8.87	8.6	8.42	8.3	8.22
	K	2.64	4.51	6.67	9.4	12.5	16.0	20.0	24.4
III	m	3.85	3.65	3.49	3.34	3.24	3.16	3.1	3.07
	m	200	66.7	38.5	26.4	21.3	18.6	16.9	16.1
	m	2.08	2.29	2.58	3.0	3.57	4.37	5.35	6.61
	m	4.18	4.55	5.08	5.96	7.15	8.55	10.4	13.2
	$-m$	1.01	1.48	1.93	2.36	2.78	3.19	3.61	4.02

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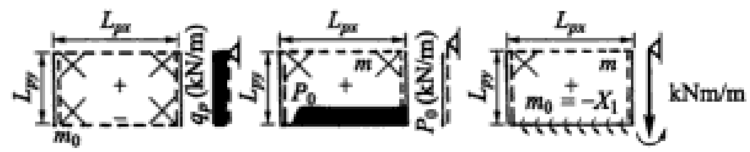
$v_1 v_j = V_i V_j / k_w$, where $K_w = V_j / w_j$, $K\theta = M_j / \theta_j$

Table 3.9. Moments, rotations and displacements for three cases.

Moment $m_i = m_{i\mathrm{II}} + m_{i\mathrm{III}} + m_{i\mathrm{I}}$
Rotation $\theta_i = \theta_{i\mathrm{I}} + \theta_{i\mathrm{II}} + \theta_{i\mathrm{III}}$
Displacement $w_i = w_{i\mathrm{I}} + w_{i\mathrm{II}} + w_{i\mathrm{III}}$



Case I Case II Case III

L_{py}/L_{px}	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
m_{xr}	4.12	4.41	4.89	5.53	6.34	7.32	8.46	9.77
$m_{xm} = q_p L_{py}^2$	7.88	8.04	8.46	9.11	9.97	11.0	12.2	13.6
m_{ym}	8.92	10.5	13.0	16.5	21.2	27.5	35.7	46.1
$m_{xye} = \pm$	2.74	3.84	5.1	6.58	8.31	10.3	12.6	15.3
$m_{xytre} = \pm$	3.83	6.32	10.1	15.8	24.5	37.6	57.2	86.5
$K_{\theta yr} = q_p L_{py}^3$	3.7	8.0	15.8	30.0	53.5	95.2	161	270
$K_{\omega r} = q_p L_{py}^4$	3.21	6.34	11.5	19.2	30.3	45.2	65.2	91.2
m_{xr}	6.9	5.6	4.9	4.5	4.3	4.2	4.1	4.1
m_{xm}	12.6	10.5	9.6	9.2	9.4	9.6	10.2	10.9
m_{ym}	200	91	52.5	40.1	33.2	29.4	26.9	25.0
$m_{x,y,e} = \pm$	4.7	5.3	6.3	7.8	9.7	12.1	15.4	20.7
$K_{\theta yr} = -P_0 L_{py}^2$	1.62	3.08	5.11	7.76	11.0	14.7	18.9	23.6
$K_{wr} = P_0 L_{py}^3$	1.86	3.42	5.55	8.68	12.8	18.7	26.7	37.2
m_{xr}	2.2	2.35	2.5	2.65	2.74	2.8	2.85	2.9
$m_{xm} = m_D$	4.6	5.7	7.9	12.5	35.0	100	0.0	−31.0
m_{ym}	2.1	2.2	2.5	3.1	4.0	5.1	6.5	8.0
$K_{\theta yr} = -m_0 L_{py}$	1.06	1.56	2.03	2.46	2.86	3.26	3.65	4.05
$K_{\omega r} = m_0 L_{py}^2$	1.66	3.13	5.13	7.69	10.9	14.6	18.9	24.0

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EXAMPLE 3.9
Using Tables 3.9 and 3.11 where relevant, compute X_1 and M_{Lm} for $L_{py}/L_{px} = 0.5$

SOLUTION
Free standing staircases
using Eq. (3.42)

$$\bar{\theta}_{\mathrm{I}} = \infty, \quad \bar{\theta}_{\mathrm{II}} = 6.76, \quad \bar{\theta}_{\mathrm{III}} = 1.93$$

$$X_1 = \frac{\dfrac{1}{24} - \dfrac{\beta}{13.52}\left(\dfrac{L_{py}}{L_2}\right)^2}{\dfrac{1}{2} + \dfrac{\beta}{1.93}\left(\dfrac{L_{py}}{L_2}\right)} q_L L_2^2 = \frac{-q_L L_2^2}{\overline{m}_{x1}}$$

$$M_{Lm} = \text{moment in the fight} = \frac{q_L L_2^2}{8} - \frac{q_L L_2^2}{\overline{m}_{x1}} = \frac{q_L L_2^2}{\overline{m}_{Lm}}$$

$\overline{m}_{Lm}$, $\overline{m}_{x1}$ for β and $\dfrac{L_{py}}{L_3}$

can be computed. The value of

$$P_0 = \frac{q_L L_3}{2}$$

ref: Tables 3.9 and 3.11.

Table 3.10. Coefficients for the determination of bending moments of staircases with two sides of the landing slabs supported.

β	L_{py}/L_3	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	m_{x1}	18.7	23.7	32.7	53.8	159	−162.0	−53.0	−31.5
	m_{Lm}	14.0	12.1	10.6	9.4	8.42	7.62	6.95	6.38
0.75	m	16.8	20.0	25.0	33.7	53.2	131.0	−261.0	−64.4
	m_{Lm}	15.3	13.3	11.8	10.5	9.42	8.52	7.76	7.12
0.5	m_{x1}	15.1	16.9	19.4	23.1	28.9	39.2	62.5	161.0
	m_{Lm}	17.1	15.2	13.6	12.2	11.1	10.0	9.17	8.42
0.25	m	13.5	14.2	15.2	16.5	18.1	20.2	23.1	17.1
	m_{Lm}	19.7	18.3	16.9	15.5	14.3	13.2	12.2	11.3

D_f = thickness of the flight slab
D_L = thickness of the landing slab

$$\beta = \left(\frac{D_f}{D_L}\right)^3$$

Table 3.11. Coefficients for the determination of bending moments of staircases with three sides of the landing slabs supported.

$$X_1 = \frac{-q_L L_3^2}{m_{x1}} \quad m_{Lm} = \frac{q_L L_3^2}{m_{Lm}}$$

β	L_{py}/L_3	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	m_{x1}	20.5	30.5	68.3	−179.0	−35.9	−19.1	−12.71	−9.31
	m_{Lm}	13.1	10.8	9.06	7.66	6.54	5.64	4.9	4.3
0.75	m_{x1}	17.9	23.6	37.1	105.0	−99.2	−31.5	−18.0	−12.3
	m_{Lm}	14.4	12.1	10.2	8.66	7.4	6.38	5.54	4.85
0.5	m_{x1}	15.7	18.6	23.9	35.4	78.9	−208.0	−41.3	21.9
	m_{Lm}	16.3	14.0	12.0	10.3	8.9	7.7	6.7	8.86
0.25	m_{x1}	13.7	14.9	16.6	19.2	23.6	31.9	53.0	2.05
	m_{Lm}	19.2	17.3	15.4	13.7	12.1	10.7	9.42	8.3

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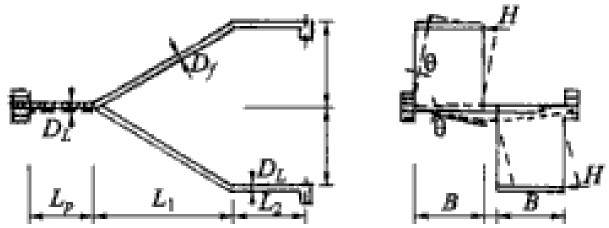
Flexibility analysis of a free standing 'scissors' type staircase

Figure 3.20 shows a typical 'scissors' type staircase. Figure 3.20 also indicates the deformation of the same staircase in plan.

The stair is idealised for the flexibility analysis. Various diagrams for m_0, m_1 and m_2 are given in Figure 3.21. Various rotations 'θ' and displacements 'w' with subscripts are given in the same figure.

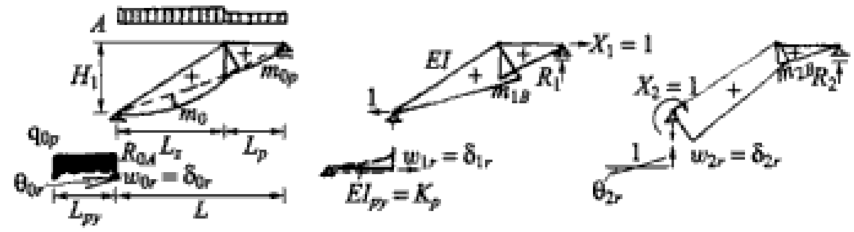
Determination of reactions and moments

The following generalised equations are derived by statics.



a) b)

Figure 3.20. Scissors type free standing stairs.



a) m_0 diagram $X_1 = X_2 = 0$ b) m_1 diagram $X_1 = 0$ c) m_2 diagram $X_2 = 1$

Figure 3.21. Flexibility diagrams.

Reactions and moments

$$R_{A0} = P_0 = \frac{qL_2}{2}\left(1 - \frac{L_2}{L_1}\right) + \frac{q_pL_p}{2}\frac{t_p}{L} \tag{3.49}$$

$$m_{0B} = \frac{q_LL_2^2}{2}\left(1 - \frac{L_2}{L}\right) + \frac{q_pL_p^2}{2}\frac{L_2}{L} \tag{3.50}$$

$$m_{0L} = \frac{q_LL_2^2}{8}, \quad m_{0p} = \frac{q_pL_p^2}{8} \tag{3.51}$$

$$K_p\theta_{0r} = q_{0p}\frac{L_{py}^3}{\bar{\theta}_{\mathrm{I}}} + p_0\frac{L_{py}^2}{\bar{\theta}_{\mathrm{II}}}, \quad K_p\delta_{0r} = q_{0p}\frac{L_{py}^4}{\overline{w}_{\mathrm{I}}} + p_0\frac{L_{py}^3}{\overline{w}_{\mathrm{II}}} \tag{3.52}$$

$$R_1 = \frac{H_1}{L}, \quad m_{1B} = R_1L_p, \quad \text{for } X_1 = 1 \tag{3.53}$$

$$R_2 = \frac{1}{L}, \quad m_{2B} = R_2L_p, \quad \text{for } X_2 = 1 \tag{3.54}$$

$$K_p\delta_{1r} = R_1\frac{L_{py}^3}{\overline{\omega}_{\mathrm{II}}}, \quad K_p\theta_{2r} = \frac{L_{py}}{\bar{\theta}_{\mathrm{III}}}, \quad K_p\delta_{2r} = R_2\frac{L_{py}^3}{w_{\mathrm{II}}} \tag{3.55}$$

Where θ_{I} and w_{I} are rotations at I point.

Flexibility coefficients

$$f_{11} = \frac{1}{3}\left(\frac{L_2}{\cos\alpha} + L_p\right)m_{1B}^2 + \beta R_1(K_p\delta_{1r}) \tag{3.56}$$

$$f_{12} = \frac{1}{6}\frac{L_2}{\cos\alpha}m_{1B}(1 + 2m_{2B}) + \frac{1}{3}L_pm_{1B}m_{2B} + \beta R_1(K_p\delta_{2r}) \tag{3.57}$$

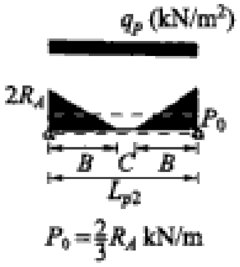


Figure 3.22. Reactions due to load q_p on landing.

$$f_{22} = \frac{1}{3}\frac{L_2}{\cos\alpha}(1 + m_{2B} + m_{2B}^2) + \frac{1}{3}L_p m_{2B}^2 + \beta(K_p\theta_{2r}) + \beta R_2(K_p\delta_{2r}) \tag{3.58}$$

$$\delta_{10} = \frac{1}{3}\frac{L_2}{\cos\alpha}m_{1B}m_{0B} + \frac{1}{3}\frac{L_2}{\cos\alpha}m_{1B}m_{0L} + \frac{1}{3}L_p m_{1B}m_{0B} + \frac{1}{3}L_p m_{1B}m_{0p} - \beta R_1(K_p\delta_{0r}) \tag{3.59}$$

$$\delta_{20} = \frac{1}{6}\frac{L_2}{\cos\alpha}m_{0B}(1 + m_{2B}) + \frac{1}{3}\frac{L_2}{\cos\alpha}(1 + m_{2B})m_{0L} + \frac{1}{3}L_p m_{2B}(m_{0B} + m_{0p}) - \beta(K_p\theta_{2r}) - \beta R_2(K_p\delta_{2r}) \tag{3.60}$$

where

$$\beta = \left(\frac{D_f}{D_L}\right)^3 \tag{3.61}$$

The flexibility matrix $[f]$ is written as:

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = -\begin{Bmatrix} \delta_{10} \\ \delta_{20} \end{Bmatrix} \tag{3.62}$$

or

$$X_1 = -\frac{f_{22}\delta_{10} - f_{12}\delta_{20}}{f_{11}f_{22} - f_{12}^2} \quad X_2 = -\frac{f_{11}\delta_{20} - f_{12}\delta_{10}}{f_{11}f_{22} - f_{12}^2} \tag{3.63}$$

The moment at B is written as:

$$M_B = m_{0B} + X_1 m_{1B} + X_2 m_{2B} \tag{3.64}$$

The axial force N is written as:

$$N_{BA} = \pm\frac{X_1}{\cos\alpha}, \quad N_{BC} = \pm X_1 \tag{3.65}$$

The load q_p gives the following reactions as shown in Figure 3.22.

3.4 FLEXIBILITY METHOD FOR HELICAL STAIRS

3.4.1 *Introduction*

A number of analyses for helical staircases were given in Chapter 2. Scordelis (1960a, b) developed an equation using the flexibility method to evaluate redundants at the mid span of helical girders when they are subjected to uniform loads. Results have been tabulated for the midspan redundants of 510 different girders with rectangular cross-sections. The variables are the horizontal angles, angle to slope and the width-depth ratio of the cross-section. Torsional effects are included. In order to bring uniformity to the text, some symbols have been changed.

3.4.2 *Notation for the analysis*

α = angle of slope
B = width
D = depth
δ_{x0} = relative linear displacement of the x-axis due to a uniform load of 1 kN per metre of horizontal projection with the redundants equal to zero
δ_{r0} = relative angular displacement about the x-axis due to a uniform load of 1 kN per metre horizontal projection with the redundants equal to zero
f_{xx} = relative linear displacement in the direction of the x-axis due to $X_x = 1$
f_{rx} = relative angular displacement about the x-axis due to $X_x = 1$
f_{xr} = relative linear displacement in the direction of the x-axis due to $X_r = 1$
f_{rr} = relative angular displacement about the x-axis due to $X_r = 1$
R = radius centre line
2ϕ = a horizontal angle
X_x = a horizontal force along and in the direction of x-axis
X_r = a moment acting about the x-axis

3.4.3 *Basic analysis*

Figures 3.23 and 3.24 give the layouts of the helical staircase with various parameters. The displacements of the redundants are written as:

$$X_x f_{xx} + X_r f_{xr} = -\delta_{x0} \tag{3.66}$$

$$X_x f_{rx} + X_r f_{rr} = -\delta_{r0} \tag{3.67}$$

Due to symmetry $f_{rx} = f_{xr}$

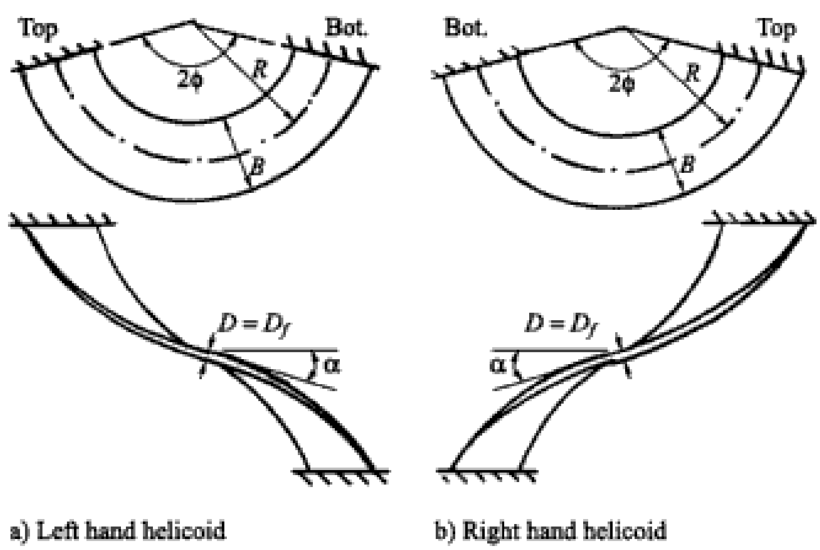


Figure 3.23. Geometry of helicoidal girders.

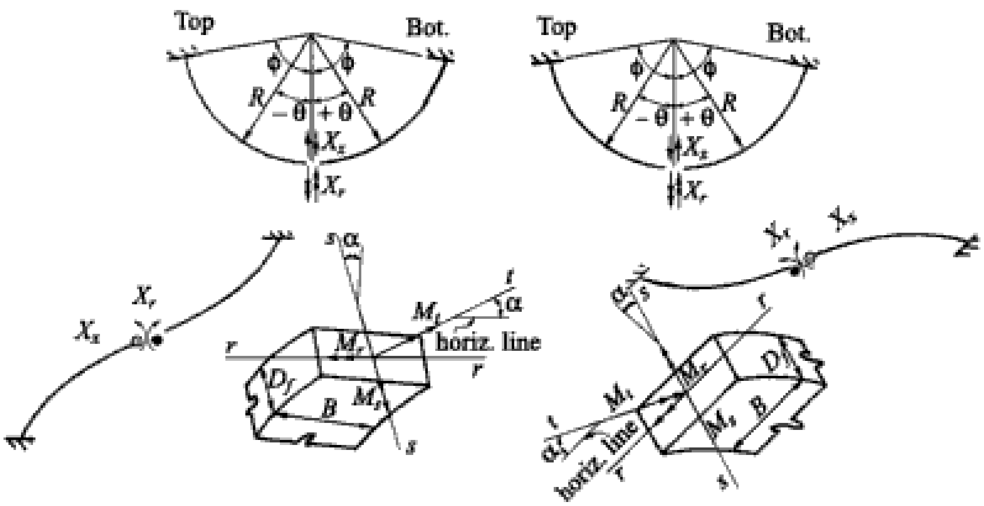


Figure 3.24. Positive directions of redundants, moments and torsion. *Note*: The moment vector is shown with a double arrowhead.

The value of δ_{x0} is written as:

$$\delta_{x0} = \int_{-\phi}^{+\phi} \frac{m_{s0} m_{r0}}{EI_r} R\,d\phi + \int_{-\phi}^{+\phi} \frac{m_{sx} m_{s0}}{EI_s} R\,d\phi + \int_{-\phi}^{+\phi} \frac{m_{sx} m_{r0}}{GJ_t} R\,d\phi \qquad (3.68)$$

where m_{r0}, m_{s0} are moments in 'r' and 's' directions, and m_{t0} in the 't' direction due to a uniform load of 1 kN/m of horizontal projections with zero indeterminacy. If θ is located at mid span, the following expressions can be written:

$$m_{r0} = -R^2(1 - \cos\theta) \qquad (3.69)$$

$$m_{s0} = -R^2(\theta - \sin\theta)\sin\alpha \qquad (3.70)$$

$$m_{t0} = -R^2(\theta - \sin\theta)\cos\alpha \qquad (3.71)$$

m_{rx}, m_{sx} and m_{tx} represent bending and torsional moments in the girder due to $X_r = 1$:

$$m_{r0} = -R(\theta\sin\theta)\tan\alpha \qquad (3.72)$$

$$m_{s0} = R(\sin\theta)\cos\alpha + R(\theta\cos\theta)\sin\alpha\tan\alpha \qquad (3.73)$$

$$m_{t0} = -R(\sin\theta)\sin\alpha + R(\theta\cos\theta)\sin\alpha \qquad (3.74)$$

m_{rr}, m_{sr} and m_{tr} represent bending and torsional moment in the girder due to $X_r = 1$:

$$m_{rr} = \cos\theta \qquad (3.75)$$

$$m_{sr} = \sin\theta\sin\alpha \qquad (3.76)$$

$$m_{tr} = \sin\theta\cos\alpha \qquad (3.77)$$

where EI_r and EI_s represent the bending stiffnesses about the r and s axes, respectively, and GJ_t represents the torsional stiffnesses, $I_t = K_1 BD^3$, the following values are to be taken:

B/D	0	1	2	4	6	8	10	12	14	16
K_1	0.1	0.15	0.223	0.256	0.295	0.31	0.32	0.322	0.325	0.327

Once X_x and X_r are evaluated from Equations (3.66) and (3.67) then Equations (3.69) to (3.77) are used to find M_r, M_s and M_t. They are given as:

$$M_r = m_{r0} + X_x m_{rx} + X_r m_{rr} \tag{3.78}$$

$$M_s = m_{s0} + X_x m_{sx} + X_r m_{sr} \tag{3.79}$$

Assuming that the helicoidal cantilever is fixed at the bottom and free at the top, Scordelis (1960a, b) developed the following expression for the displacements at the top:

$$\begin{aligned}\delta_{x0} = {} & \frac{R^4 \tan\alpha \sec\alpha}{EI_r}\left[\sin\phi - \phi\cos\phi - \frac{1}{8}D\right] \\ & + \frac{R^4 \sec\alpha \sin\alpha}{EI_s} E' \cos\alpha \\ & - \left[2\phi\cos\phi + (\phi^2 - 2)\sin\phi - \overline{D}\tan\alpha\sin\alpha\right] \\ & + \frac{R^4 \sin\alpha}{GJ_t}\left[(3 - \phi^2)\sin\phi - 3\phi\cos\phi \right. \\ & \qquad \left. - \frac{\phi}{2} + \frac{3}{8}\sin 2\phi - \frac{\phi\cos 2\phi}{4}\right]\end{aligned} \tag{3.80}$$

$$\begin{aligned}\delta_{r0} = {} & \frac{R^4 \sec\alpha}{EI_r}\left[\frac{\phi}{2} - \sin\phi + \frac{\sin 2\phi}{4}\right] + \frac{R^3 \sin^2\alpha \sec\alpha}{EI_s}\overline{E} \\ & + \frac{R^3 \cos\alpha}{GJ_t}\overline{E}\end{aligned} \tag{3.81}$$

$$\begin{aligned}f_{xx} = {} & \frac{R^3 \tan^2\alpha \sec\alpha}{EI_r}\left[\overline{E} - \frac{\phi\cos 2\phi}{4}\right] \\ & + \frac{R^3 \sec\alpha}{EI_s}\left\{\overline{H}\cos^2\alpha + \frac{1}{4}\overline{D}\sin^2\alpha \right. \\ & \qquad \left. + \left[\overline{F} + \frac{\phi\cos 2\phi}{4}\right]\tan^2\alpha \sin^2\alpha\right\} \\ & + \frac{R^3 \sec\alpha \sin^2\alpha}{GJ_t}\left\{\overline{H} - \frac{1}{4}\overline{D} + \left[\overline{F}\sin 2\phi + \frac{\phi\cos 2\phi}{4}\right]\right\}\end{aligned} \tag{3.82}$$

$$f_{rr} = \frac{R\sec\alpha}{EI_r}\overline{G} + \frac{R\sec\alpha\sin^3\alpha}{EI_s}\overline{H} + \frac{R\cos\alpha}{GJ_t}\overline{H} \tag{3.83}$$

$$\begin{aligned}f_{rx} = f_{xr} = {} & \frac{R^2 \sec\alpha \tan\alpha}{EI_r}\overline{D} \\ & + \frac{R^2 \sin\alpha \sec\alpha}{EI_s}\left[\overline{H}\cos\alpha + \overline{D}\tan\alpha\sin\alpha\right] \\ & + \frac{R^2 \sin\alpha}{GJ_t}\left[-\overline{H} + \frac{1}{8}(\overline{D})\right]\end{aligned} \tag{3.84}$$

where

$$\overline{D} = (\sin 2\phi - 2\phi \cos 2\phi)$$

$$\overline{E} = \left[\frac{\phi}{2} - \sin\phi + \phi\cos\phi - \frac{\sin 2\phi}{4}\right]$$

$$\overline{F} = \frac{\phi^3}{6} + \left(\frac{\phi^2}{4} - \frac{1}{8}\right)\sin 2\phi$$

$$\overline{G} = \frac{\phi}{2} + \frac{\sin 2\phi}{4}$$

$$\overline{H} = \frac{\phi}{2} - \frac{\sin 2\phi}{4}$$

Scordelis (1960a, b) gives results in terms of X_r/R^2, X_x/R, M_r/R^2, M_s/R^2, and M_t/R^2 for various values of ϕ ranging from 0 deg to 300 deg. Based on these equations, the author extended the results up to 360 deg. They are given in Figures 3.24 to 3.28 for various values of B/D and α. The load is assumed to be 1 kN/m. When the total load due to dead and imposed loads is known, the values from these figures are modified by multiplying respective values such as X_r/R^2 etc. by that load.

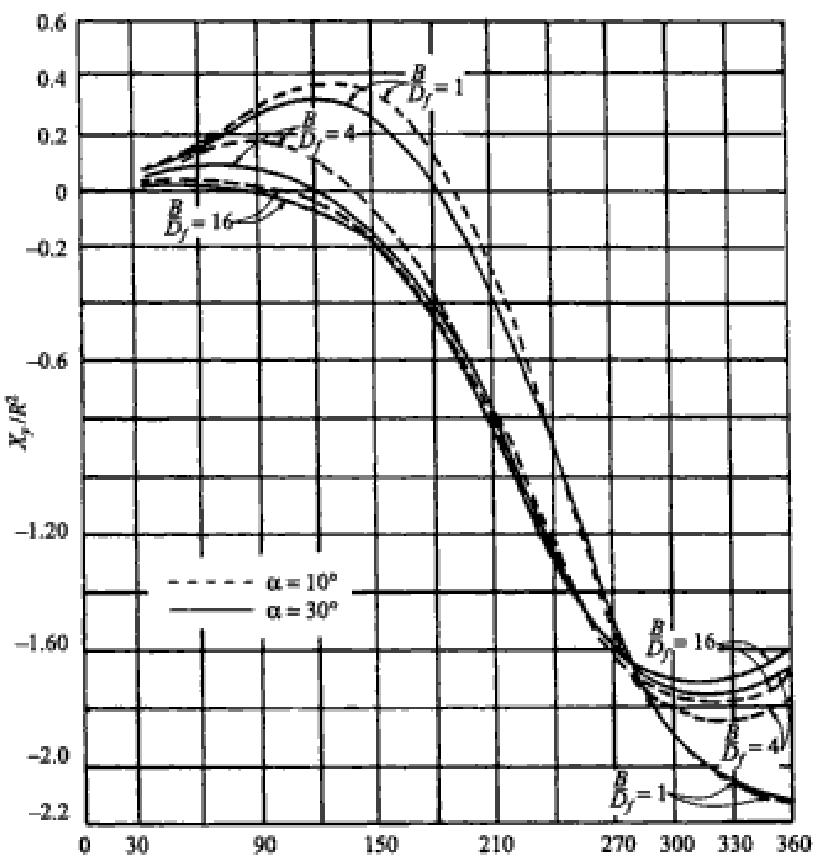


Figure 3.25. Values of maximum bending moment. M_{r1} for girders of various horizontal angles ϕ.

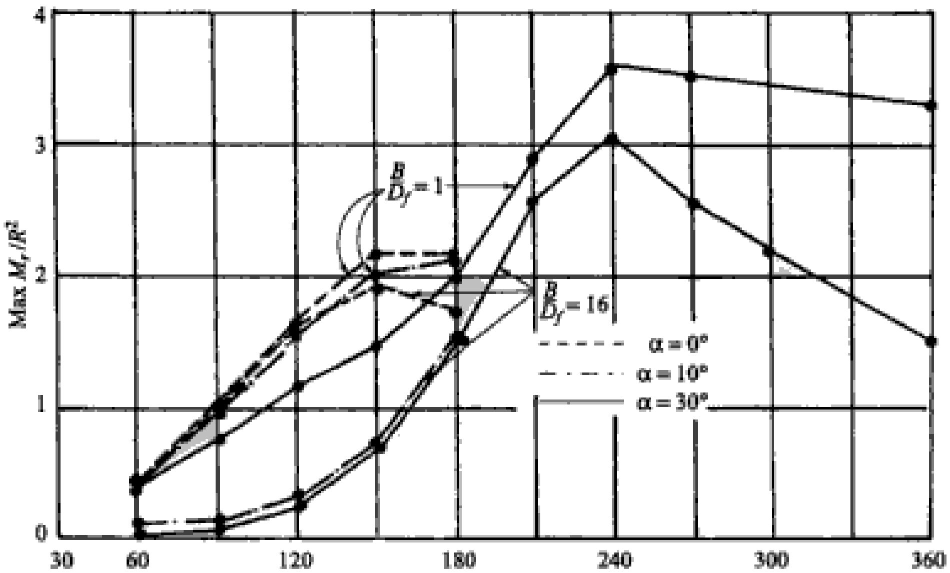


Figure 3.26. Values of maximum bending moment M_r.

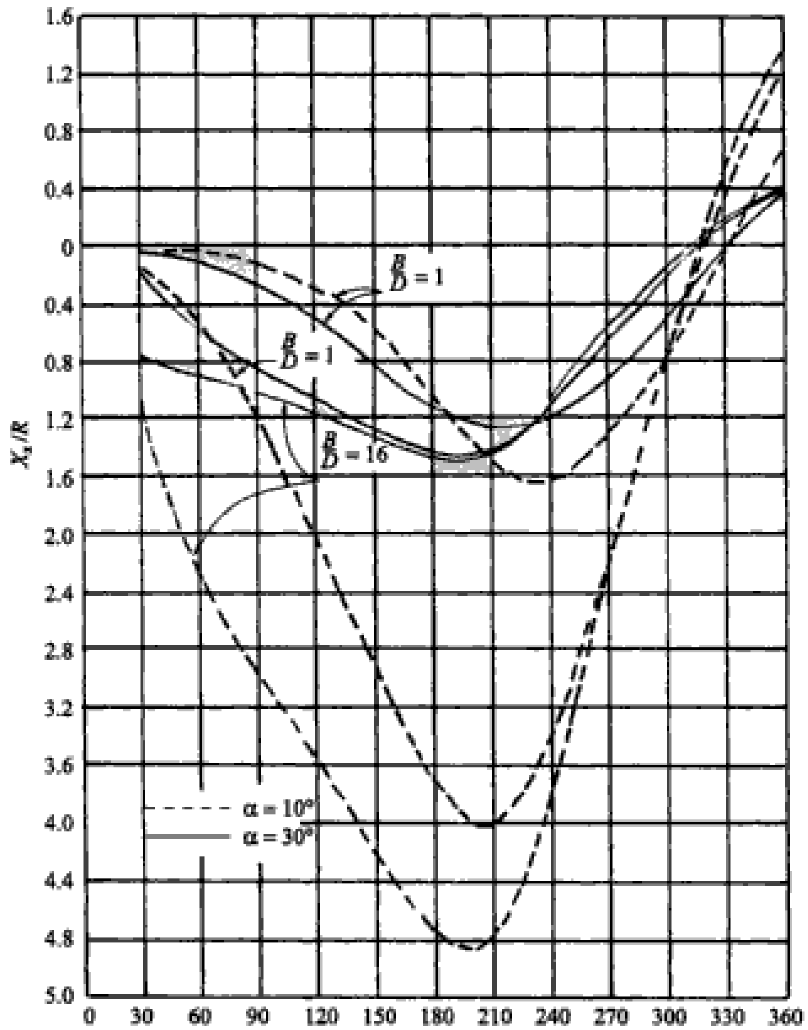


Figure 3.27. Horizontal angle values of redundant X_x.

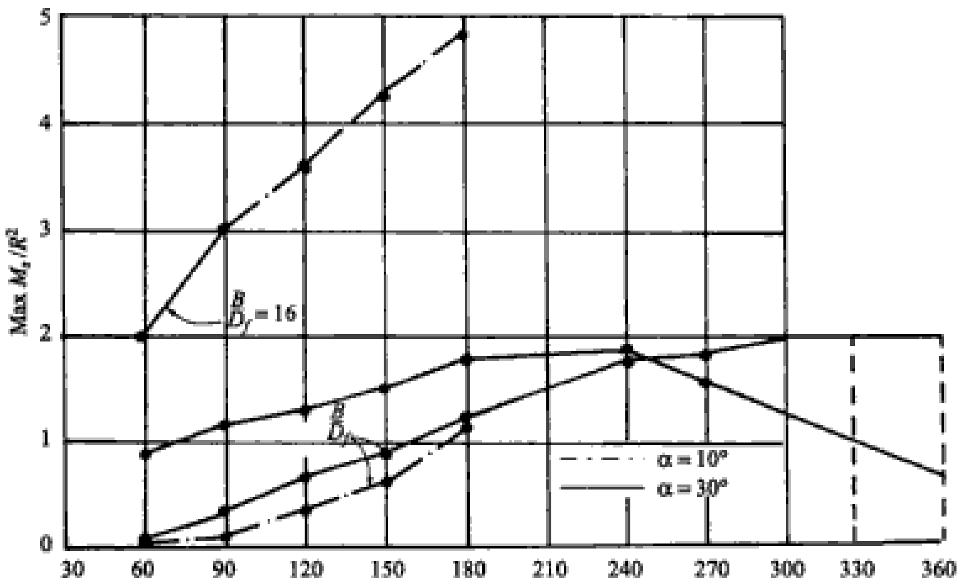


Figure 3.28. Horizontal angle values of maximum bending moment M_s.

EXAMPLE 3.10
Analyse the helical staircase using the flexibility method and the following data:

$$B/D_f = 16$$
$$G/E = 0.7$$
$$\alpha = 30$$
$$w = \text{uniform load} = 5.205 \text{ kN/m}$$
$$\phi = 0 \text{ to } 270$$

SOLUTION
Flexibility of analysing helical staircase

The above values are based on the load with a magnitude of 1 kN/m. For a common dead and imposed load of 5.205 kN/m the above values in the table in brackets are arrived at using the above figures multiplied by 5.205 kN/m.

For example, when $R = 25$ m and $\phi_f = 90$ and $w = 5.205$ kN/m

$M_r/R^2 = 0.781$ $M_r = (25)^2 \times 0.781 = 488.125$ kN/m
$M_s/R^2 = 6.767$ $M_s = (25)^2 \times 6.767 = 4229.375$ kN/m
$M_t/R^2 = 0.052$ $M_t = (25)^2 \times 0.052 = 32.5$ kN/m

ϕ	30°	90°	120°	150°	180°	210°	240°	270°
X_s/R	0.78 (4.06)	1.0 (5.205)	1.0 (5.205)	1.17 (6.089)	1.56 (8.12)	1.40 (7.287)	1.18 (6.142)	0.6 (3.123)
X_r/R^2	0.04 (0.208)	0 (0)	−0.07 (−0.365)	−0.20 (−1.041)	−0.45 (−2.343)	−0.88 (−4.580)	−1.30 (−6.767)	−1.65 (8.588)
M_r/R^2	0 (0)	0.15 (0.781)	0.5 (2.603)	0.88 (4.580)	1.50 (7.81)	2.45 (12.753)	3.10 (16.136)	2.5 (13.013)
M_s/R^2	(0) 0	1.30 (6.767)	1.40 (7.287)	1.60 (8.328)	1.80 (9.369	1.85 (9.629)	1.90 (9.890)	1.70 (8.849)
M_t/R^2	0 (0)	0.01 (0.052)	0.01 (0.052)	0.26 (1.354)	0.50 (2.603)	1.40 (7.287)	2.65 (6.897)	3.80 (19.78)

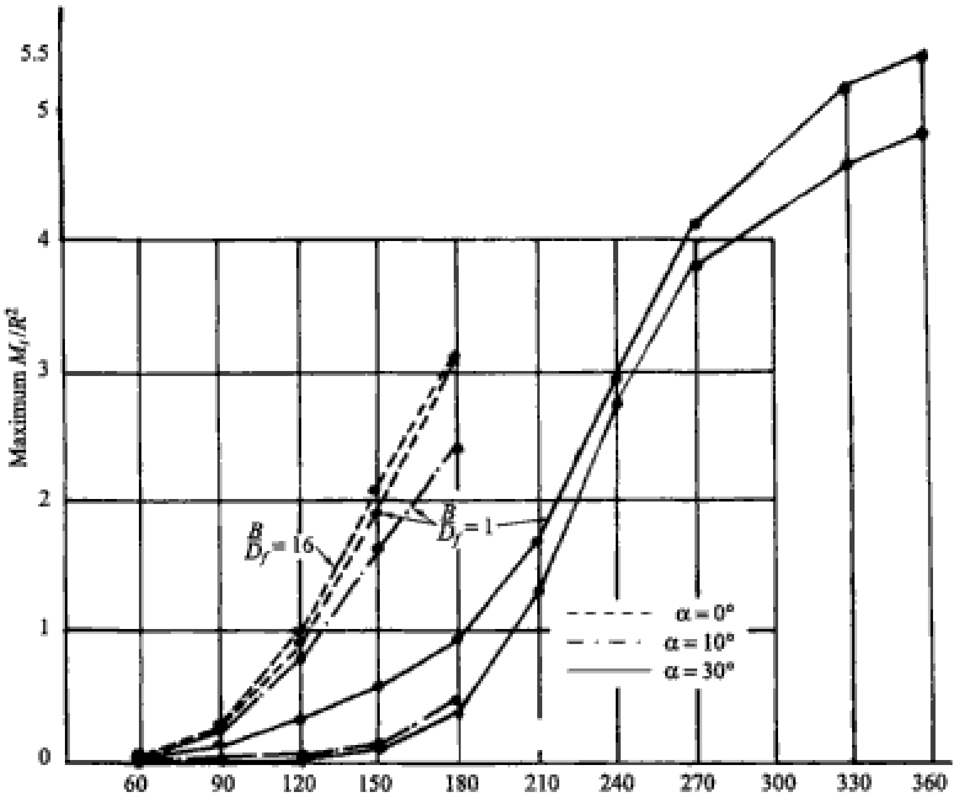


Figure 3.29. M_t/R^2 versus ϕ.

3.5 MEMBRANE PLATE/SHELL ANALYSIS

3.5.1 *Introduction*

Helical stairs can be treated as a plate or shell and hence a membrane theory can be considered for computing forces. A simple method would be to assume that they are axisymmetric. A polar co-ordinate system is then adopted. Figure 3.30(a) shows a helical staircase and Figure 3.30(b) shows the usual forces on interior and exterior surfaces.

3.5.2 *Notation for the analysis*

r_s = radius to the outside of the stairs
$r_r = \rho r_s$ = radius to the inside of the stairs
$B = \beta r_s$ = width of the stairs
$\xi = r/r_s$ = parameter
H = overall height

Parameters, $k = H/2\pi r_s$, $\eta = \sqrt{\xi^2 + k^2}$
γ_m = radius to the interior $= r_s(1+\rho)/2$
N_ϕ, N_r and $N_{r\phi}$ = forces as shown on surfaces
u, v, w = displacements in ϕ, r and normal directions
q_L = load per unit area

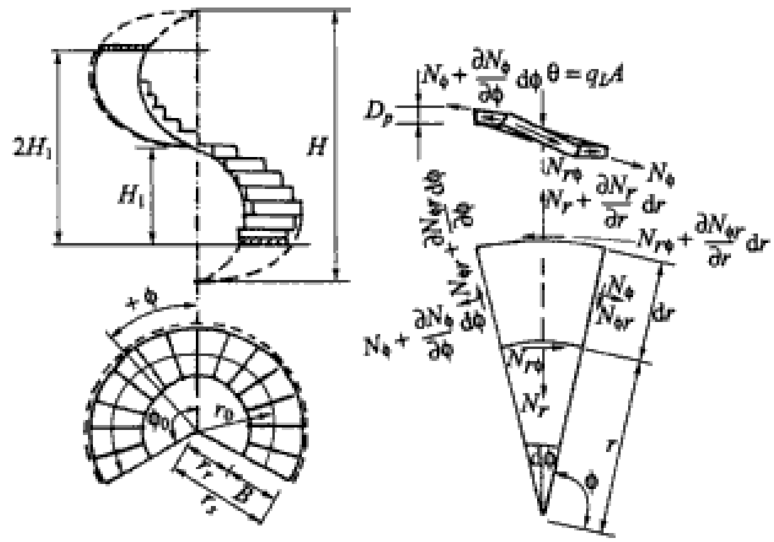


Figure 3.30. Helical stair membrane analysis.

a) Helical stair plan and elevation   b) Forces on helical stairs

3.5.3 *Basic analysis*

The equilibrium equations are summarised for an element of the stair surfaces:

$$\frac{2K}{r_s}\left(\frac{N_{r\phi}}{\xi^2}\right) + q_L = 0$$

$$\xi^2 \frac{\partial N\phi}{\partial \phi} + \eta \frac{\partial\left(N_{r\phi}\xi^2\right)}{\partial \xi} = 0$$

$$\frac{\partial(\eta N_r)}{\partial \xi} + \frac{\partial N_{r\phi}}{\partial \phi} - \frac{\xi}{\eta} N_\phi = 0 \tag{3.85}$$

The forces acting on the stair are written as:

$$N_\phi = \frac{E D_f}{r_s}\left(\frac{1}{\eta}\frac{\partial u}{\partial \phi} + \frac{\xi}{\eta^2} v\right)$$

$$N_r = \frac{E}{r_s}\frac{\partial u}{\phi} \tag{3.86}$$

$$N_{r\phi} = \frac{E D_f}{2 r_s}\left(\frac{\partial u}{\partial \xi} - \frac{\xi}{\eta^2} u + \frac{1}{\eta}\frac{\partial u}{\partial \phi} - \frac{2k}{\eta^2} w\right)$$

where E is the Young's modulus and D_f is the stair thickness.

The following equations for displacements can then easily be derived:

$$
\begin{aligned}
u &= -\frac{q_0 r_s^2}{48 E D_f k}\bar{c}\phi^2\left[\xi f(\xi) - 4\eta^2\left(\frac{1}{\xi} - 1\right)\right] \\
v &= \frac{q_0 r_s^2}{24 E D_f k}\bar{c}\phi\eta f(\xi) \\
w &= \frac{q_0 r_s^2}{96 E D_f k^2}\bar{c} \\
&\quad \times \left\{2\eta^2 f(\xi) + \frac{8}{3}\eta^2\left[3 - 2\xi - \frac{\rho^2}{\xi^2}(1+2\beta)\right.\right. \\
&\qquad \left.\left. - \phi^2\left[k^2 f(\xi) + \xi\eta^2\frac{\partial f}{\partial \xi} + 4\eta^2\left(\xi + \frac{k^2}{\xi^2}\right)\right]\right]\right\}
\end{aligned}
\tag{3.87}
$$

where the function $f(\xi)$ and $\bar{c}$ are defined as:

$$
\begin{aligned}
f(\xi) &= 4 - \xi - \frac{1}{\eta}\sqrt{1+k^2} + \frac{1}{\eta}\left[2\rho(1+\beta) - k^2\right]\ln\frac{1+\sqrt{1+k^2}}{\xi+\eta} \\
\bar{c} &= \frac{\beta^2(1+2\rho)}{\ln\frac{1}{\rho} - \frac{\beta}{6}(6+3\beta+2\beta^2)} \\
&= \frac{1+2\rho}{\beta^2\left(1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots\right)}
\end{aligned}
\tag{3.88}
$$

where ln = natural log.
Using Eq. (3.46), the forces are written as:

$$
\begin{aligned}
N_\phi &= \frac{q_0 r_s}{6}\bar{c}\phi\eta\left(\frac{1}{\xi} - 1\right) \\
N_r &= -\frac{q_0 r_s}{12k}\frac{\bar{c}\phi}{\eta}\left[\xi^2 - 2\xi + \rho(1+\beta)\right] \\
N_{r\phi} &= -\frac{q_0 r_s}{36k}\bar{c}\left[3 - 2\xi - \frac{\rho^2}{\xi^2}(1+2\beta)\right]
\end{aligned}
\tag{3.89}
$$

For $\xi = 1$, u and v are zero and for $\xi = \rho$, the new values of N_r and N_ϕ from Eq. (3.89) are calculated. Putting Eq. (3.89) into Eq. (3.85), then expression for 'q_L' is derived.

$$
w = q_L = \frac{q_0}{18}\bar{c}\left[\frac{3}{\xi^2} - \frac{2}{\xi} - \frac{\rho^2}{\xi^4}(1+2\beta)\right]
\tag{3.90}
$$

N (total) considering top and bottom:

$$
N_{r\phi} = -\frac{q_0 r_s^2}{18k}\bar{c}\beta^3
\tag{3.91}
$$

The horizontal and vertical components of N_ϕ are:

$$N_{\phi h} = \frac{q_0 r_s}{6k}\bar{c}\phi(1-\xi) \tag{3.92}$$

$$N_{\phi V} = \frac{q_0 r_s}{6k}\bar{c}\phi\left(\frac{1}{\xi}-1\right) \tag{3.93}$$

and their respective total values are given below:

$$N_{\phi h\text{total}} = \frac{q_0 r_s^2}{12k}\bar{c}\phi_0\beta^2 \tag{3.94}$$

$$r_{\phi y} = \frac{\beta^2}{2\left(\ln\frac{1}{\rho}-\beta\right)}r_s \tag{3.95}$$

$$N_{\phi V\text{total}} = \frac{q_0 r_s^2}{6}\bar{c}\phi_0\left(\ln\frac{1}{\rho}-\beta\right) \tag{3.96}$$

Helical stairs having sectors subtended $2\phi_0$ in plan.

For a sector of $2\phi_0$ in plan (see Fig. 3.30) of the helical stairs, Equations (3.89) and (3.91) are modified.

$$N_\phi = -\frac{q_0 r_s}{6k}\bar{c}\phi\eta\left(\frac{1}{\xi}-1\right)$$

$$N_r = \frac{q_0 r_s}{12k}\bar{c}\frac{\phi}{\eta}\left[\xi^2-2\xi+\rho(1+\beta)\right] \tag{3.97}$$

$$N_{r\phi} = \frac{q_0 r_s}{36k}\bar{c}\left[3-2\xi-\frac{\rho^2}{\xi^2}(1+2\beta)\right]$$

EXAMPLE 3.11

Using the following data, calculate the geometrical parameters and the forces in a helical staircase made of reinforced concrete.

The helical staircase plan is shown in Figure 3.31.

Data

$H = 3$ m

$n = 17$

h_1 = height of the step = 0.1765

$\overline{G}$ = going = 0.3 m

General

$r_s = 3$ m $\quad B = 1.5$ m $\quad r_r = 1.5$ m

$r_m = 2.25$ m $\quad q_0 = 9.49$ kN/m^2 (vertical load)

$\overline{C} = 18.9$ exterior $\quad \bar{c} = 56.8$ interior

(respective factors times dead + imposed)

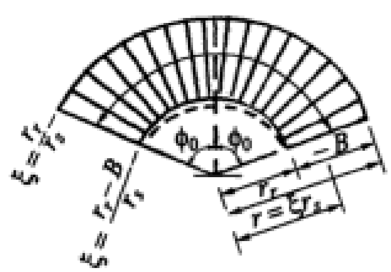


Figure 3.31. Helical stairs in plan with total suspended angle $2\phi_0$.

SOLUTION

Plate/shell analysis using membrane theory for a helical staircase.

Parameters:

$2\phi_0$ = total angle subtended in plan

$\phi_0 = 65°$

$$H = \frac{180}{\phi_0} 2H_l = \frac{180}{65} \times 3 = 8.3 \text{ m}$$

for stair slope $\tan\alpha = \dfrac{3.0}{17 \times 0.3} = 0.59$

$\cos\alpha = 0.86$

Outside of the stair

$$\rho = \frac{r_r}{r_s} = 0.5, \quad \beta = \frac{B}{r_s} = 0.5, \quad k = \frac{H}{2\pi r_s} = 0.44$$

$$\eta = \sqrt{\xi^2 + k^2} = \sqrt{\xi^2 + 0.194}$$

Adopting Eq. (3.97)

$$\left.\begin{aligned} N_\phi &= 7.15 q_0 r_s \phi \sqrt{\xi^2 + 0.194}\left(\frac{1}{\xi} - 1\right) \\ N_r &= -3.58 q_D r_s \frac{\phi}{\sqrt{\xi^2 + 0.194}}(\xi^2 - 2\xi + 0.75) \\ N_{r\phi} &= -1.199 q_D r_s \left(3 - 2\xi - \frac{1}{2\xi^2}\right) \end{aligned}\right] \text{Range } 0.5 < \xi < 1.0$$

for $\xi = \rho = 0.5$

$N_\phi = -4.76 q_D r_s \phi$ where ϕ is a particular value of ϕ_0

$N_r = 0$

$N_{r\phi} = 0$

$$q_L(\xi) = 1.05 q_0 \left(\frac{3}{0.5^2} - \frac{2}{0.5} - \frac{1}{2(0.5)^4}\right) = (0) q_0 = 0$$

for $\xi = 0.75$

$N_\phi = 2.88 q_0 r_s \phi$

$N_r = -0.775 q_0 r_s \phi$

$N_{r\phi} = +0.725 q_0 r_s$

$q_L(\xi) = 1.1445 q_0$

for outside $r_s = 3.0$ m

$q_0 = 9.49$ kN/m^2

$\phi = \phi_0 = 65°$

for $\xi = 1.0$

$N_\phi = (0)(q_0 r_s \phi) = 0$

$N_r = -0.820 q_0 r_s \phi$

$N_{r\phi} = +0.60 q_0 r_s$

$q_L(\xi) = 0.525 q_0$

These results are left in $r_s\phi$ and q_0. All other values can be interpolated. The plate bending, if any, can be computed from a general equation.

$$m_b = \frac{q_0 B^2}{32} = \frac{q_0 B^2 r_s^2}{32} = \frac{9.49(1.5)^2}{32} = 0.67 \text{ kN m/m}$$

$N_{r\phi\text{total}} = -25.5$ kN $\quad N_{\phi r\text{total}} = 87.0$ kN

$N_{\text{total}} = 59.0$ kN $\quad N_{\phi h\text{total}} = 87.0$ kN

For the inside of the stair

$$r_s = 1.5 \text{ m}, \quad B = -1.50 \text{ m}, \quad r_r = 3.0 \text{ m}, \quad \rho = \frac{r_r}{r_s} = \frac{3.0}{1.5} = 2.0$$

$$\beta = \frac{B}{r_s} = -1 \quad k = 0.88 \quad \eta = \sqrt{\xi^2 + 0.775} \quad \bar{c} = 56.8$$

$$\left.\begin{aligned}
N_\phi &= 10.76 q_0 r_s \phi \sqrt{\xi^2 + 0.775}\left(\frac{1}{\xi} - 1\right) \\
N_r &= 5.38 q_0 r_s \frac{\phi}{\sqrt{\xi^2 + 0.775}}(\xi^2 - 2\xi) \\
N_{r\phi} &= 1.79 q_0 r_s \left(3 - 2\xi + \frac{4}{\xi^2}\right) \\
q(\xi) &= 3.15 q_0 \left(\frac{3}{\xi^2} - \frac{2}{\xi^2} + \frac{4}{\xi^4}\right)
\end{aligned}\right] \text{Range } 1 \leqslant \xi \leqslant 2$$

$\xi = 1$

$N_\phi = 0$

$N_r = 4.04 \times (q_0 r_s \phi)$

$N_{r\phi} = -8.95 q_0 r$

$q(\xi) = -4.75 q_0$

$\xi = 2$

$N_\phi = -11.85 q_0 r_s \phi$

$N_r = 0$

$N_{r\phi} = 0$

$q(\xi) = 1.0 q_0$

3.5.4 *Helical stairs with torsion included using the flexibility method of analysis*

Table 3.12 gives a general analysis of an element of the helical staircase under bending, shear and torsion. This analysis acts as the basis of the proposed flexibility method of analysis.

The general $[f_{ij}]_g$ including torsion is written as:

$$[f_{ij}]_g = \sum_s \left[\frac{m_{yi}m_{yj}}{EI_y} + \frac{m_{zi}m_{zj}}{EI_z} + \frac{m_{ti}m_{tj}}{GI_t} + x_{vy}\frac{v_{yi}v_{yj}}{GA} + x_{vz}\frac{v_{zi}v_{zj}}{GA} + \frac{n_i n_j}{EA} \right] \mathrm{d}s \tag{3.98}$$

In pure bending, the first three terms of Eq. (3.98) apply

$$\{\delta_{i0}\}_g = \sum_s \begin{bmatrix} \dfrac{m_{yi}m_{y0}}{EI_y} + \dfrac{m_{zi}m_{z0}}{EI_z} + \dfrac{m_{ti}m_{t0}}{GI_t} \quad (3.99) \\ + \\ x_{vy}\dfrac{v_{yi}v_{y0}}{GA} + x_{vz}\dfrac{v_{zi}v_{z0}}{GA} + \dfrac{n_i n_0}{EA} \quad (3.100) \end{bmatrix} \mathrm{d}s$$

$$\mathrm{d}s = \frac{r\,\mathrm{d}\phi}{\cos\alpha} \quad \text{(reference Table 3.12)} \tag{3.101}$$

Where I_y, I_z, I_t are along respective axes and where there are x parameters for shear such as a shape factor:

$$GI_T = 2E\frac{I_y I_z}{I_y + I_z} \tag{3.102}$$

$$[f_{ij}] = \frac{\cos\alpha EI_y}{r}[f_{ij}]_g \tag{3.103}$$

For a pure bending case Eq. (3.103) can be expressed as:

$$f_{ij} = \sum \left[m_{yi}m_{yj} + \frac{I_y}{I_z}m_{zi}m_{zj} + \frac{1}{2}\left(1 + \frac{I_y}{I_z}\right)m_{ti}m_{tj} \right] \mathrm{d}\phi \tag{3.104}$$

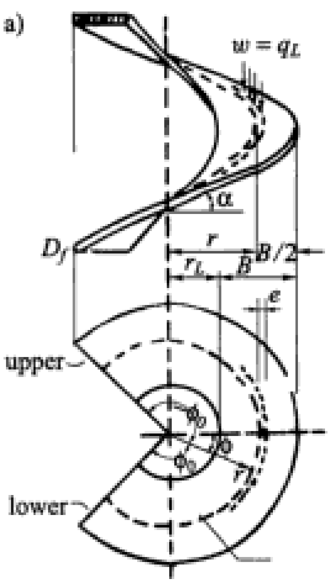
When a redundant $X_i = 0$, only loading exists

$$\begin{aligned} m_{y0} &= f(\phi) \\ m_{z0} &= \sin\alpha \left[f'(\phi) + r^2 \int_0^\phi q_L \,\mathrm{d}\phi \right] \\ m_{t0} &= -\cos\alpha \left[f'(\phi) + r^2 \int_0^\phi q_L \,\mathrm{d}\phi \right] \end{aligned} \tag{3.105}$$

$$
\begin{aligned}
&m_{y1} = 0, && m_{y4} = \sin\phi \\
&m_{z1} = 1, && m_{z4} = \sin\alpha\cos\phi \\
&m_{t1} = 0, && m_{t4} = -\cos\alpha\cos\phi \\
&m_{y2} = 0, && m_{y5} = \tan\alpha\phi\sin\phi \\
&m_{z2} = \cos^2\alpha, && m_{z5} = -(\sin\alpha\tan\alpha\phi\cos\alpha \\
&m_{t2} = \sin\alpha\cos\alpha, && \qquad\qquad + \cos\alpha\sin\phi) \\
&m_{y3} = \tan\alpha\phi\cos\phi, && m_{t5} = \sin\alpha(\phi\cos\phi - \sin\phi) \\
&m_{z3} = -(\sin\alpha\tan\alpha\phi\sin\phi && m_{y6} = \cos\phi \\
&\qquad\qquad - \cos\alpha\cos\phi && m_{z6} = -\sin\alpha\sin\phi \\
&m_{t3} = \sin\alpha(\phi\sin\phi + \cos\phi), && m_{t6} = \cos\alpha\sin\phi
\end{aligned}
\tag{3.106}
$$

Table 3.12. Basic analysis for helical stairs.

Figure (a) shows an elevation and a plan for a helical staircase with a sectoral plan.
Figure (b) gives the position of forces on an element. This is modified by including load w and other components in Figure b). 2 and Figure b). 3.



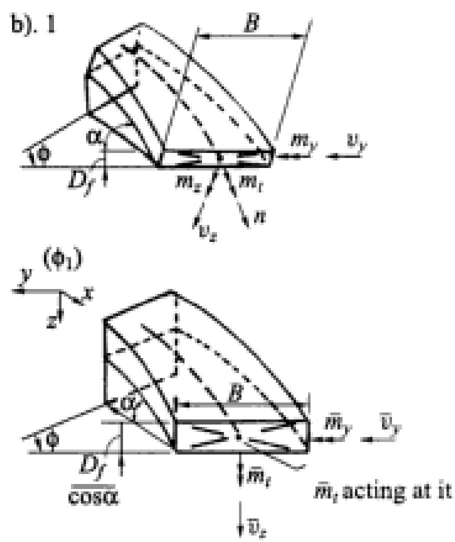


Figure (a). Helical stair – elevation and plan.

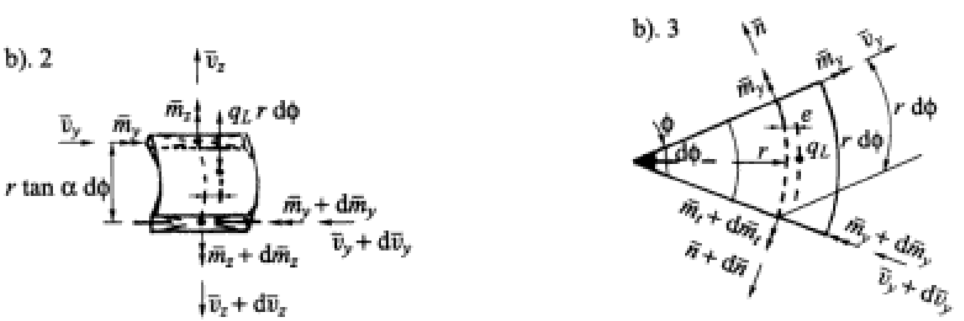


Figure (b). Forces on the stair element.

Table 3.12 (cont.).

General equations

Moments, shears and axial loads

$$\begin{aligned} &\bar{n} = n\cos\alpha - v_z\sin\alpha, && n = \bar{v}\sin\alpha + \bar{n}\cos\alpha \\ &\bar{v}_y = v_y, && v_y = \bar{v}_y \\ &\bar{v}_z = n\sin\alpha v_z\cos\alpha, && v_z = \bar{v}_z\cos\alpha - \bar{n}\sin\alpha \\ &\bar{m}_t = m_t\cos\alpha - m_z\sin\alpha, && m_t = \bar{m}_z\sin\alpha + \bar{m}_t\cos\alpha \\ &\bar{m}_y = m_y, && m_y = \bar{m}_y \\ &\bar{m}_z = m_t\sin\alpha + m_z\cos\alpha, && m_z = \bar{m}_z\cos\alpha - \bar{m}_t\sin\alpha \end{aligned} \tag{A}$$

Equilibrium equations

The following equations can easily be derived:

Forces:

$$\begin{aligned} &\bar{n} + \frac{\partial\bar{v}_y}{\partial\phi} = 0 && \text{radial} \\ &\frac{\partial\bar{n}}{\partial\phi} - \bar{v}_y = 0 && \text{tangential} \\ &q_L r + \frac{\partial\bar{v}_y}{\partial\phi} = 0 && \text{vertical} \end{aligned} \tag{B}$$

moments:

$$\begin{aligned} &r\bar{v}_z - r\tan\alpha\bar{n} - \bar{m}_t = \frac{\partial\bar{m}_y}{\partial\phi} && \text{radial} \\ &q_L r_e + r\tan\alpha\bar{v}_y - \frac{\partial\bar{m}_t}{\partial\phi} + \bar{m}_y = 0 && \text{tangential} \\ &r\bar{v}_y + \frac{\partial\bar{m}_t}{\partial\phi} = 0 && \text{vertical} \end{aligned} \tag{C}$$

The values $\bar{v}$, $\bar{n}$ and $\bar{m}$ can now be computed using the following equations derived from Equations (B) and (C).
Note: x, a redundant values $= X$ introduced in the flexibility analysis

$$\begin{aligned} &\bar{v}_y = \frac{1}{r}(x_5\cos\phi + x_3\sin\phi) \\ &\bar{v}_z = -r\int_0^\phi q_L\,\mathrm{d}\phi - \frac{1}{r}\sin\alpha x_1 \\ &\bar{n} = \frac{1}{r}(x_5\sin\phi + x_3\cos\phi) \\ &\bar{m}_y = x_4\sin\alpha + x_6\cos\phi + \tan\alpha(x_3\phi\cos\phi - x_5\phi\sin\phi) + f(x) \\ &\bar{m}_z = \cos\alpha(x_1 + x_2) - x_5\sin\phi + x_3\cos\phi \\ &\bar{m}_t = -r^2\int_0^\phi q_L\,\mathrm{d}\phi - x_4\cos\phi + x_6\sin\phi - \sin\alpha x_1 - f'(\phi) + \tan\alpha(x_3\phi\sin\phi + x_5\phi\cos\phi) \end{aligned} \tag{D}$$

The value $f(\phi)$ when determined from differentiation is given as

$$f''(\phi) + f(\phi) = q_L r(r + e) \tag{E}$$

When $\phi = 0$ in Eq. (D), the forces and moments in the helical staircase can be written as:

$$\begin{aligned} &\bar{v}_y(0) = \frac{1}{r}x_5, && \bar{m}_y(0) = f(0) + x_6 \\ &\bar{v}_z(0) = -\frac{1}{r}\sin\alpha x_1, && \bar{m}_z(0) = \cos\alpha(x_1 + x_2) + x_3 \\ &\bar{n}(0) = -\frac{1}{r}x_3, && \bar{m}_t(0) = -x_4 + \sin\alpha x_1 - f'(0) \end{aligned} \tag{F}$$

where x's are indeterminacies as X in the main flexibility analysis.

Table 3.12 (cont.).

Substituting Eq. (A) into Eq. (D), the following values are obtained:

$$
\begin{aligned}
v_y &= \frac{1}{r}(x_5 \cos\phi + x_3 \sin\phi) \\
v_z &= -r\cos\alpha \int_0^{\phi} q_L \, d\phi - \frac{1}{r}\sin\alpha\cos\alpha x_1 - \frac{1}{r}\sin\alpha(x_5 \sin\phi - x_3\cos\phi) \\
n &= -r\sin\alpha \int_0^{\phi} q_L \, d\phi - \frac{1}{r}\sin^2\alpha x_1 + \frac{1}{r}\cos\alpha(x_5\sin\phi - x_3\cos\phi) \\
m_y &= f(\phi) + \tan\alpha(x_3\phi\cos\phi - x_5\phi\sin\phi) + x_4\sin\phi + x_6\cos\phi \\
m_z &= \sin\alpha\left[f'(\phi) + r^2\int_0^{\phi} q_L \, d\phi + x_4\cos\phi - x_6\sin\phi - \tan\alpha(x_3\phi\sin\phi + x_5\phi\cos\phi)\right] \\
&\quad - \cos\alpha(x_5\sin\phi - x_3\cos\phi) + x_1 + x_2\cos^2\alpha \\
m_t &= -\cos\alpha\left[f'(\phi) + r^2\int_0^{\phi} q_L \, d\phi - \sin\alpha(x_2) + x_4\cos\phi - x_6\sin\phi\right] \\
&\quad + \sin\alpha[x_3\phi\sin\phi + x_5\phi\cos\phi + x_3\cos\phi - x_5\sin\phi]
\end{aligned}
\tag{G}
$$

When one edge is fixed and the other free as shown in Figure (c) Eq. (F) can be written as:

$$
\begin{aligned}
&x_1 = x_2 = x_3 = x_5 = 0 \\
&x_4 = -f'(0) \\
&x_6 = -f(0) \\
&q_L = B(g_k + q_k) = w \\
&r_e = r + e = \frac{2}{3}\frac{r_a^3 - r_i^3}{r_a^2 - r_i^2} = r + \frac{B^2}{12r}
\end{aligned}
\tag{H}
$$

$$f(\phi) = -qr^2 ke \quad \text{and} \quad f'(\phi) = 0 \tag{I}$$

$$ke = 1 + \frac{e}{r} \tag{J}$$

Using Equations (H) to (J) and substituting Eq. (G) into (J), the following equations are finally derived:

$$
\begin{aligned}
v_y &= 0 \\
v_z &= -q_L r\cos\alpha(\phi) \\
n &= -q_L r\sin\alpha(\phi) \\
m_y &= -q_L r^2 ke(1 - \cos\phi)
\end{aligned}
\tag{K}
$$

$$
\begin{aligned}
m_z &= q_L r^2 \sin\alpha(\phi - ke\sin\phi) \\
m_t &= q_L r^2 \cos\alpha(\phi - ke\sin\phi)
\end{aligned}
\tag{K1}
$$

Linear relation

$$
\begin{aligned}
&\phi = ZB \\
&q_L(\phi) = q_{L0} + \frac{q_{\phi 0} - q_{L0}}{q_{L0}}\phi = k_1 + k_2\phi \\
&f(\phi) = -(k_1 + k_2\phi)r^2 ke \\
&f'(\phi) = -k_2 r^2 ke
\end{aligned}
\tag{L}
$$

Table 3.12 (cont.).

Equation (L) is substituted into Eq. (G) for the additional values.

c)

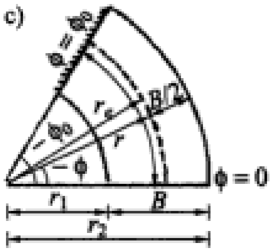


Figure (c). Dimensional parameters.

The values of various moments can be computed between $-\phi_0$ to $+\phi_0$. Various flexibility coefficients are evaluated using Table 3.13.

$$\left.\begin{aligned} f_{11} &= 2\beta'\phi_0 \\ f_{12} &= 2\beta'\cos^2\alpha\phi_0 \\ f_{13} &= 2\beta'\left[\cos\alpha\sin\phi_0 + \sin\alpha\tan\alpha(\phi_0\cos\phi_0 - \sin\phi_0)\right] \\ f_{14} &= 2\beta'\sin\alpha\sin\phi_0 \\ f_{15} &= f_{16} = 0 \end{aligned}\right\} \tag{3.107}$$

$$\begin{aligned} f_{22} &= \cos^2\alpha(2\overline{F} - 1 + 3\beta')\phi_0 \\ f_{23} &= \cos\alpha\left[(2\overline{F} - 1 + 2\beta')(2\sin\phi_0 - \phi_0\cos\phi_0) + \beta'\phi_0\cos\phi_0\right] \\ f_{24} &= -\sin\alpha\cos^2\alpha(1 - \beta')\sin\phi_0 \\ f_{25} &= 0 = f_{26} \end{aligned}$$

$$\begin{aligned} f_{33} = \tan^2\alpha\Big[&\frac{1}{3}(2 - \overline{F})\phi_0^3 + \frac{1}{2}\overline{F}\phi_0^2\sin 2\phi_0 \\ &+ \frac{1}{4}(2 - 3\bar{f} - 2\beta)(\sin 2\phi_0 - 2\phi_0\cos 2\phi_0)\Big] \\ &+ \frac{1}{4}(2\overline{F} - 1 + 3\beta')(2\phi_0 + \sin 2\phi_0) \end{aligned}$$

$$\begin{aligned} f_{35} &= f_{36} = 0 \\ f_{45} &= f_{46} = 0 \\ f_{34} &= \frac{1}{4}\tan\alpha\Big[\overline{F}(\sin 2\phi_0 - 2\phi_0\cos 2\phi_0) \\ &\qquad - 2(1 - \overline{F} - \beta')(2\phi_0 + \sin 2\phi_0)\Big] \\ f_{44} &= (2 - \overline{F})\phi_0 - \frac{1}{2}\overline{F}\sin 2\phi_0 \end{aligned}$$

$$f_{55} = \tan^2\alpha\left[\frac{1}{3}(2-\overline{F})\phi_0^3 - \frac{1}{2}\overline{F}\phi_0^2\sin 2\phi_0 - \frac{1}{4}(2-3\overline{F}-2\beta')(\sin 2\phi_0 - 2\phi_0\cos 2\phi_0)\right] + \frac{1}{4}(2\overline{F}-1+3\beta')(2\phi_0 - \sin 2\phi_0)$$

$$f_{66} = (2-\overline{F})\phi_0 + \frac{1}{2}\overline{F}\sin 2\phi_0$$

$$f_{56} = -\frac{1}{4}\tan\alpha\left[2(1-\overline{F}-\beta')(2\phi_0 - \sin 2\phi_0) + \overline{F}(\sin 2\phi_0 - 2\phi_0\cos 2\phi_0)\right]$$

where

$$\beta' = \frac{I_y}{I_z}$$

Table 3.13. Tabulated values of ϕ's versus θ's versus M's.

	$\theta_y = 0$	$\theta_y = 1/3$	$y = \alpha$	
$-\phi_0$	−0.250 (−10.8)	−0.200 (−8.76)	0.500 (21.6)	$M_y/(wr^2)$
$-\phi_0/2$	0.045 (1.944)	0.050 (2.16)	0.120 (5.184)	
0	−0.110 (−4.752)	−0.158 (−6.83)	−0.230 (−9.94)	
$+\phi_0/2$	0.045 (1.944)	0.050 (2.16)	0.100 (432)	
$+\phi_0$	−0.30 (−12.96)	−0.20 (−8.76)	0.000 (0)	
$-\phi_0$	1.200 (−5.184)	−1.250 (−54)	−1.300 (−56.16)	$M_z/(wr^2)$
$-\phi_0/2$	−1.250 (−54)	−1.250 (−54)	−1.300 (56.16)	
0	0.000 (0)	0.000 (0)	0.000 (0)	
$+\phi_0/2$	1.250 (54)	1.250 (54)	1.400 (60.48)	
$+\phi_0$	1.200 (57.84)	1.250 (54)	1.400 (60.48)	
$-\phi_0$	−0.055 (−2.376)	−0.125 (−54)	−0.200 (−8.76)	$M_t/(wr^2)$
$-\phi_0/2$	0.035 (−1.512)	−0.010 (−0.432)	0.025 (1.08)	
0	0.000 (0)	0.000 (0)	0.000 (0)	
$+\phi_0/2$	8.035 (1.512)	0.010 (0.432)	−0.025 (−1.08)	
$+\phi_0$	0.055 (−2.376)	0.125 (54)	0.180 (7.78)	

$wr^2 = 43.2$ kN m.

$$\overline{F} = \left(1 - \frac{1}{2}\cos^2\alpha\right)(1-\beta) \tag{3.108}$$

Similarly

$$\partial_{j0} = \sum_{-\phi_0}^{+\phi_0}\left[m_{yj}m_{y0} + \beta m_{zj}m_{z0} + \frac{1}{2}(1+\beta)m_{tj}m_{t0}\right]\mathrm{d}\phi \tag{3.109}$$

Hence six equations are written and solved

$$\begin{aligned} &f_{11}x_1 + f_{12}x_2 + \cdots f_{16}x_6 + \delta_{10} = 0\\ &f_{21}x_1 + f_{22}x_2 + \cdots f_{26}x_6 + \delta_{20} = 0\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &f_{61}x_1 + f_{62}x_2 + \cdots f_{66}x_6 + \delta_{60} = 0 \end{aligned} \tag{3.110}$$

The symmetric and un-symmetric matrices are written as:

symmetric

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14}\\ f_{21} & f_{22} & f_{23} & f_{24}\\ f_{31} & f_{32} & f_{33} & f_{34}\\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{Bmatrix} = -\begin{Bmatrix} \delta_{10}\\ \delta_{20}\\ \delta_{30}\\ \delta_{40} \end{Bmatrix} \tag{3.111}$$

un-symmetric

$$\begin{bmatrix} f_{55} & f_{56}\\ f_{65} & f_{66} \end{bmatrix}\begin{Bmatrix} x_5\\ x_6 \end{Bmatrix} = -\begin{Bmatrix} \delta_{50}\\ \delta_{60} \end{Bmatrix}$$

Since $f(-\phi)$ as functions

$$\text{load } w = q_L = B(g_k + q_k) = \text{constant} \tag{3.112}$$

$$f(\phi) = -q_L r^2 ke \quad \text{and} \quad f'(\phi) = 0$$

where

$$\frac{e}{r} = \frac{B^2}{12r^2}$$

$$ke = \left(1 + \frac{e}{r}\right) \tag{3.113}$$

$$m_{y0} = -q_L r^2 ke \qquad \text{(a)}$$

$$m_{z0} = q_L r^2 \sin\alpha(\phi) \qquad \text{(b)}$$

$$m_{t0} = -q_L r^2 \cos\alpha(\phi) \qquad \text{(c)(3.114)}$$

$$\delta_{10} = \delta_{20} = \delta_{30} = \delta_{40} = 0 \qquad \text{(d)}$$

$$\delta_{50} = 2q_L r^2 \tan\alpha\big[(3 + ke - 3\overline{F} - \beta')(\sin\phi_0 - \phi_0\cos\phi_0) - (1-\overline{F})\phi_0^2\sin\phi_0\big] \qquad \text{(e)}$$

$$\delta_{60} = 2q_L r^2\big[(1-\overline{F})\phi_0\cos\phi_0 - (1 + ke - \overline{F})\sin\phi_0\big] \qquad \text{(f)}$$

If

$$\overline{X}_5 = \frac{x_5}{q_L r^2}, \quad \overline{X}_6 = \frac{x_6}{q_L r^2}$$

then moments are written as: shears and axial thursts due to vertical loads as

$$v_y = q_L r \overline{X}_5 \cos\phi \qquad \text{(a)}$$

$$v_z = -q_L r\left(\cos\alpha\phi + \overline{X}_5 \sin\alpha \sin\phi\right) \qquad \text{(b)}$$

$$n = -q_L r\left(\sin\alpha\phi - \overline{X}_5 \cos\alpha \sin\phi\right) \qquad \text{(c)}$$

$$m_y = -q_L r^2\left(ke + \overline{X}_5 \tan\alpha(\phi\sin\phi) - \overline{X}_6 \cos\phi\right) \qquad \text{(d)}$$

$$\begin{aligned} m_z = -q_L r^2\Big[&\left(\overline{X}_5 \cos\alpha + \overline{X}_6 \sin\alpha\right)\sin\phi - \sin\alpha\phi \\ &+ \overline{X}_5 \sin\alpha \tan\alpha(\phi\cos\phi)\Big] \qquad \text{(e)} \end{aligned} \qquad (3.115)$$

$$\begin{aligned} m_t = -q_L r^2\Big[&\left(\overline{X}_5 \sin\alpha - \overline{X}_6 \cos\alpha\right)\sin\phi + \cos\alpha\phi \\ &- \overline{X}_5 \sin\alpha(\phi\cos\phi)\Big] \qquad \text{(f)} \end{aligned}$$

These moments, shear and axial effects are also shown in Figure 3.32.

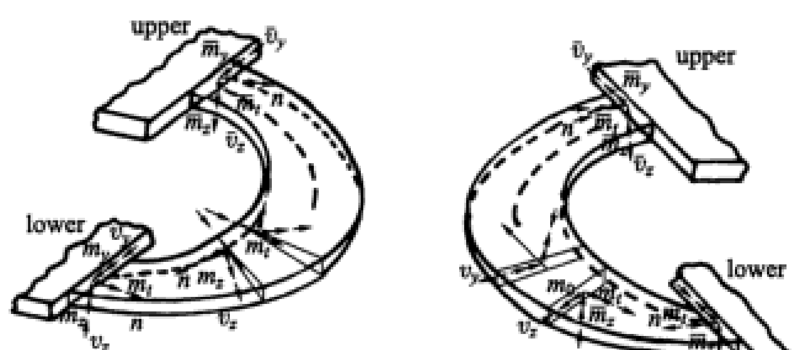


Figure 3.32. Demonstration of helical stairs of moments, shear and axial effects.

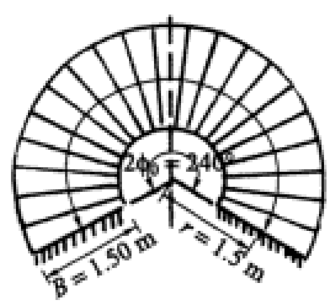


Figure 3.33. A helical staircase in plan.

EXAMPLE 3.12

A helical stairs with 240° sector in plan is shown in Figure 3.33. Using the following parameters, determine the shears and axial load for the stairs:

$2H_c$ = 3.5 m

RC slab thickness = D_f = 150 mm

n = 21

Slab width = B = 1.5 m

$h_1 = 0.1667$ m

r_i = r = 1.5 m

$\overline{G}$ = 0.2992 m

r_m = 2.25 m

$2\phi_0$ = 240°

$1.4g_k$ = characteristic imposed load = 7.80 kN/m^2

$1.6q_k$ = characteristic imposed load = 5.00 kN/m^2

q_L = 12.80 kN/m^2

w = Bq_L

q_L = 1.5 × 12.80 = 19.2 kN/m^2

SOLUTION

A helical stair with 240°: sector in plan

Parameters:

$$\beta' = \frac{I_y}{I_z} = \left(\frac{D_f}{B}\right)^2 = \left(\frac{0.15}{1.50}\right)^2 = 0.01$$

$$\alpha = \tan^{-1}\left(\frac{0.1667}{0.2992}\right) = 29°$$

$$\sin\alpha = 0.448$$

$$\cos\alpha = 0.874$$

$$\tan\alpha = 0.557$$

$$\overline{F} = \left[1 - \frac{1}{2}(0.874)^2(1.0 - 0.01)\right] = 0.612$$

$$ke = \frac{1 + (1.5)^2}{12(1.5)^2} = 1.083$$

flexibility coefficients:

$$f_{55} = 1.9866 + 2.0412\theta_y$$

$$f_{56} = -(0.6368 - 1.0103\theta_y)$$

$$f_{66} = 2.6420 + 0.5\theta_y$$

$$\delta_{50} = (3.1258 + 2.1882\theta_y)q_L r^2$$

$$\delta_{60} = -(3.3606 - 1.083)q_L r^2$$

for $q_L r^2 = 1$

$$\overline{X}_5 = -\frac{6.1184 + 11.4290\theta_y}{4.8431 + 7.6729\theta_y}$$

$$\overline{X}_6 = \frac{4.6857 + 6.4728\theta_y}{4.8431 + 7.6729\theta_y}$$

$$L_y = 3.5 \text{ m}, \quad \bar{\theta}_y = 6$$

then

$$\theta = \left(\frac{EI_y}{EI_z}\right)\frac{L_y}{\bar{\theta}r}\cos\alpha = \frac{1}{3}$$

substituting into $\overline{X}_5$ and $\overline{X}_6$ equations

$$\text{for } \theta_y = \frac{1}{3} \quad \overline{X}_5 = -1.341 \quad \overline{X}_6 = 0.925$$

$$\text{for } \theta_y = 0 \quad \overline{X}_5 = -1.261 \quad \overline{X}_6 = 0.968$$

$$\text{for } \theta_y = \infty \quad \overline{X}_5 = -1.490 \quad \overline{X}_6 = 0.844$$

for various values of θ against the values of $\theta_y = 0$, $^1/_3$ and ∞, the partial factor for $M_y/(wr^2)$, $M_z/(wr^2)$ and $M_t/(wr^2)$ can be evaluated easily using the basic theory described earlier.

Similarly the values of $V_y/(wr)$, $V_z/(wr)$, $N_t/(wr)$ can be determined and they are given in Table 3.14.

For example, the value for $wr = 19.2 \times 1.5 = 28.8$ kN and $wr^2 = 19.2 \times (1.5)^2 = 43.2$ kN. The values in the above tables are modified and are written in brackets. Bending moments and shear forces and axial thrusts are drawn for $2\phi = 240°$. They will show where and how much reinforcement is required.

Table 3.14. Numerical values of V_z/wr for various ϕ's.

$-\phi_0$	$-\phi_0/2$	0	$+\phi_0/2$	$+\phi_0$	
0.671	−0.671	−1.346	−0.671	0.671	V_y/wr = values
(19.3)	(−19.3)	(−38.77)	(−19.3)	(19.3)	wr = 28.8 kN
1.266	0.351	0	−0.351	−1.266	V_z/wr = values
(36.96)	(10.12)	(0)	(−10.12)	(−36.46)	wr = 28.8 kN
2.033	1.524	0	−1.524	−2.033	N/wr = values
(58.55)	(43.90)	(0)	(−43.90)	(−58.55)	wr = 28.8 kN

3.5.5 *Helical stairs with a horseshoe shape in plan*

A typical helical staircase has been analysed with a sectoral shape in plan. This work is extended by extending the edges by an inclined length L to form a horseshoe which is shown in Figure 3.34. All symbols are consistent with the previous analyses.

It is assumed that one edge at the support is fixed.

$$v_{y0} = 0 \tag{a}$$

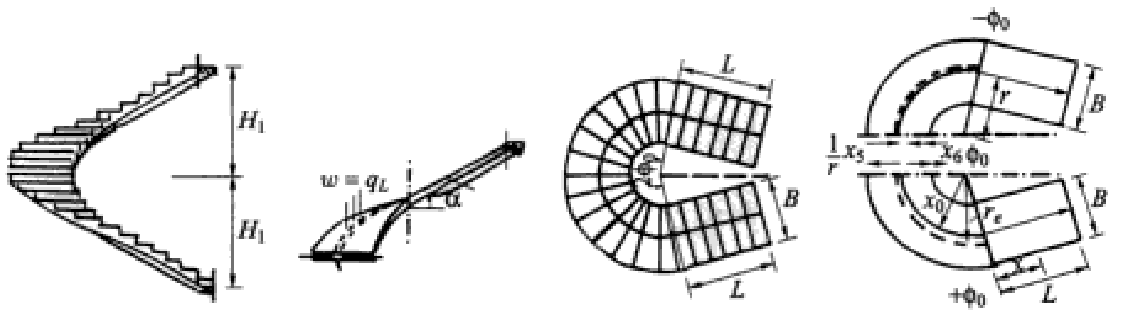
$$v_{z0} = -qr\cos\alpha\phi \tag{b}$$

$$n_0 = -qr\sin\alpha \tag{c}$$

$$m_{y0} = -qr^2\sin ke(1-\cos\phi) \qquad \text{(d) (3.116)}$$

$$m_{z0} = -qr^2\sin\alpha(\phi - ke\sin\phi) \tag{e}$$

$$m_{t0} = -qr^2\cos\alpha(\phi - ke\sin\phi) \tag{f}$$



a) Elevation b) Plan c) Parameters

Figure 3.34. A horseshoe type helical stair.

Equation (G) in Table 3.12 is invoked when $X_5 = 1$ and $X_6 = 1$

$X_5 = 1$

$$v_{y5} = \frac{1}{r}\cos\phi \quad \text{(a)}$$
$$v_{z5} = -\frac{1}{r}\sin\alpha\sin\phi \quad \text{(b)}$$
$$n_5 = \frac{1}{r}\cos\alpha\sin\phi \quad \text{(c)} \quad (3.116a)$$
$$m_{y5} = -\tan\alpha\phi\sin\phi \quad \text{(d)}$$
$$m_{z5} = -(\cos\alpha\sin\phi + \tan\alpha\sin\alpha\phi\cos\phi) \quad \text{(e)}$$
$$m_{t5} = -\sin\alpha(\sin\phi - \phi\cos\phi) \quad \text{(f)}$$

$X_6 = 1$

$$v_{y6} = v_{z6} = n_6 = 0 \quad \text{(a)}$$
$$m_{y6} = \cos\phi \quad \text{(b)}$$
$$m_{z6} = -\sin\alpha\sin\phi \quad \text{(c)} \quad (3.116b)$$
$$m_{t6} = \cos\alpha\sin\phi \quad \text{(d)}$$

Where the value of the load q'_L is included for upper and lower extended posts of length L of the horseshoe and $\phi = \phi_0$.

$$v_y = 0 \quad \text{(a)}$$
$$v_{z0} = \mp\left[q_L r\cos\alpha\phi_0 + q'_L\cos\alpha(x)\right] \quad \text{(b)}$$
$$n_0 = +q_L r\sin\alpha(\phi_0) + q'_L\sin\alpha(x) \quad \text{(c)}$$
$$m_{y0} = -q_L r^2\left[ke(1-\cos\phi_0) + \frac{\phi_0}{r}x\right] - q'_L\frac{x^2}{2} \quad \text{(d)} \quad (3.117)$$
$$m_{z0} = \pm q_L r^2\sin\alpha[\phi_0 - ke\sin\phi_0] \quad \text{(e)}$$
$$m_{t0} = \mp q_L r^2\cos\alpha[\phi_0 - ke\sin\phi_0] \quad \text{(f)}$$

$$X_5 = 1$$

$$v = \frac{1}{r}\cos\phi_0 \qquad \text{(a)}$$

$$v_{z5} = \mp\frac{1}{r}\sin\alpha\sin\phi_0 \qquad \text{(b)}$$

$$n_5 = \pm\frac{1}{r}\cos\alpha\sin\phi_0 \qquad \text{(c)}$$

$$m_{y5} = -\tan\alpha\phi_0\sin\phi_0 - \frac{1}{r}\sin\alpha\sin\phi_0(x) \qquad \text{(d)} \quad (3.118)$$

$$m_{z5} = \mp\left[\cos\alpha\sin\phi_0 + \frac{\sin^2\alpha}{\cos\alpha}\phi_0\cos\phi_0 + \frac{\cos\phi_0}{r\cos\alpha}(x)\right] \qquad \text{(e)}$$

$$m_{t5} = \mp\sin\alpha(\sin\phi_0 - \phi_0\cos\phi_0) \qquad \text{(f)}$$

$$X_6 = 1$$

$$v_{y5} = v_{z6} = n_6 = 0 \qquad \text{(a)}$$

$$m_{y6} = \cos\phi_0 \qquad \text{(b)}$$

$$m_{z6} = \mp\sin\alpha\sin\phi_0 \qquad \text{(c)} \quad (3.118a)$$

$$m_{t6} = \pm\cos\alpha\sin\phi_0 \qquad \text{(d)}$$

Again writing the generalised equations for moments where the extended parts of length 'L' are included

$$f_{ij} = \int_s \left[\frac{m_{yi}m_{yj}}{EI_y} + \frac{m_{zi}m_{zj}}{EI_z} + \frac{m_{ti}m_{tj}}{GI_t}\right]\mathrm{d}s \qquad (3.119)$$

For the two limits of $\phi = 0$ to ϕ_0 and $x = 1$ to L

$$\delta_{ij} = \int_{\phi=0}^{\phi}\left[m_{yi}m_{yj} + \frac{I_y}{I_z}m_{zi}m_{zj} + \frac{1}{2}\left(1 + \frac{I_y}{I_z}\right)m_{ti}m_{tj}\right]\mathrm{d}\phi \qquad (3.120)$$

$$+ \int_{x=0}^{L}\left[m_{yi}m_{yj} + \frac{I_y}{I_z}m_{zi}m_{zj} + \frac{1}{2}\left(1 + \frac{I_y}{I_z}\right)m_{ti}m_{tj}\right]\frac{\mathrm{d}x}{r} \qquad (3.121)$$

where

$$GI_t = \frac{2EI_Y I_Z}{(I_y + I_z)}$$

All other coefficients can be determined easily using a normal procedure. The complicated ones such as f_{55}, f_{56}, f_{66}, δ_{50}, δ_{60} need to be evaluated.

It is important to note that as usual $\beta = I_y/I_z$ and $\rho = L/r$:

$$f_{55} = a_7 \tan^2\alpha + \frac{1+\beta}{2}\sin^2\alpha(\rho a_2^2 + a_3 - 2a_5 + a_8)$$
$$+ \rho\tan^2\alpha\sin^2\phi_0\left[\phi_0(\phi_0+\rho) + \frac{\rho^2}{3}\cos^2\alpha\right]$$
$$+ \beta\Big[\cos^2\alpha a_3 + a_5(2\sin^2\alpha) + a_8\sin^2\alpha\tan^2\alpha$$
$$+ \rho\big(\cos^2\alpha\sin^2\phi_0 + \sin^2\alpha\tan^2\alpha(\phi_0^2 + \sin\phi_0\cos\phi_0)\big)$$
$$+ \rho^2\cos\phi_0\big(\sin\phi_0 + \tan^2\alpha(\phi_0\cos\phi_0)\big)$$
$$+ \frac{\rho}{3}\frac{\cos^2\phi_0}{\cos^2\alpha}\Big] \tag{3.122}$$

$$f_{56} = -a_5\tan\alpha - \frac{1+\beta}{4}\sin 2\alpha(\rho a_2\sin\phi_0 + a_3 - a_5)$$
$$- \frac{\rho}{2}\tan\alpha\sin 2\phi_0\left(\phi_0 + \frac{\rho}{2}\cos\alpha\right)$$
$$+ \beta\sin\alpha\Big[(a_3 + \tan^2 a_5)\cos\alpha + \rho\cos\alpha\sin\phi_0$$
$$\times\left(\sin\phi_0 + \tan^2\alpha(\phi_0\cos\phi_0) + \frac{\rho\cos\phi_0}{2\cos^2\alpha}\right)\Big] \tag{3.122a}$$

$$f_{56} = a_4 + \frac{1+\beta}{2}\cos^2\alpha\big(a_3 + \rho\sin^2\phi_0\big) + \rho\cos^2\phi_0$$
$$+ \beta\sin^2\alpha\big(a_3 + \rho\sin^2\phi_0\big) \tag{3.122b}$$

$$\delta_{50} = q_L r^2\left\{\begin{aligned} & ke(a_2 - a_5)\tan\alpha + \rho^2\sin\alpha\sin\phi_0\left[\frac{ke}{2}s^{\mathrm{I}} + s^{\mathrm{IV}}\right] \\ & + \rho\tan\alpha\phi_0\sin\phi_0\big[s^{\mathrm{I}} + s^{\mathrm{III}}\big] \\ & + \frac{1+\beta}{4}\sin 2\alpha\big[a_2 - a_6 + \rho a_2 s^{\mathrm{II}} + ke(a_5 - a_3)\big] \end{aligned}\right\}$$
$$- \beta q r^2\sin\alpha\Big[\cos\alpha(a_2 - kea_3) + \sin\alpha\tan\alpha(a_6 - kea_5)$$
$$+ \rho s^{\mathrm{II}}\Big(\cos\alpha\sin\phi_0 + \frac{\sin^2\alpha}{\cos\alpha}\phi_0\cos\phi_0$$
$$+ \frac{\rho\cos\phi_0}{2\cos\alpha}\Big)\Big] \tag{3.123}$$

$$\delta_{60} = -q_L r^2 \left\{ kea_1 + \frac{1+\beta}{2}\cos^2\alpha\left[a_3 - kea_3 + \rho\sin\phi_0(s^{\mathrm{II}})\right] + \rho\cos\phi_0\left[ke(1-\cos\phi_0) + s^{\mathrm{III}}\right]\right\} - \beta q r^2 \sin^2\alpha\left[a_2 - kea_3 + \rho\sin\phi_0(s^{\mathrm{II}})\right] \tag{3.124}$$

where,

$$a_1 = \sin\phi_0 - \frac{1}{2}(\phi_0 + \sin\phi_0\cos\phi_0) \tag{a}$$

$$a_2 = \sin\phi_0 - \phi_0\cos\phi_0 \tag{b}$$

$$a_3 = \frac{1}{2}(\phi_0 - \sin\phi_0\cos\phi_0) \tag{c}$$

$$a_4 = \frac{1}{2}(\phi_0 + \sin\phi_0\cos\phi_0) \tag{d}$$

$$a_5 = \frac{1}{2}(\phi_0\sin^2\phi_0) - \frac{1}{4}(\phi_0 - \sin\phi_0\cos\phi_0) \tag{e}$$

$$a_6 = \phi_0^2\sin\phi_0 - 2(\sin\phi_0 - \phi_0\cos\phi_0) \tag{f}$$

$$a_7 = \frac{\phi_0}{6} - \frac{\phi_0^2}{2}\sin\phi_0\cos\phi_0 + \frac{1}{2}\left(\phi_0\sin^2\phi_0 + \frac{1}{2}\left\{\phi_0\sin^2\phi_0 - \frac{1}{2}(\phi_0 - \sin\phi_0\cos\phi_0)\right\}\right) \tag{g}$$

$$a_8 = \frac{\phi_0^3}{6} + \frac{\phi_0^2}{2}\sin\phi_0\cos\phi_0 - \frac{1}{2}(\phi_0\sin^2\phi_0) - \frac{1}{4}(\phi_0 - \sin\phi_0\cos\phi_0) \tag{h}$$

$$s^{\mathrm{I}} = 1 - \cos\phi_0 \tag{i}$$

$$s^{\mathrm{II}} = \phi_0 - ke\sin\phi_0 \tag{j}$$

$$s^{\mathrm{III}} = \phi_0\frac{\rho}{2} + \frac{\rho^2}{6} \tag{k}$$

$$s^{\mathrm{IV}} = \phi_0\frac{\rho}{3} + \frac{\rho^2}{8} \tag{l}$$

The values of coefficients a_1 to a_8 have been (other values can be interpolated) tabulated below in Table 3.15 after solving various integrals, for various values of ϕ_s.

Similarly for the helical and horseshoe stairs, the following values are derived.

In the lower and upper lengths L of the horseshoe

Table 3.15. Parametric values.

ϕ_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0	0	0	0	0	0	0	0	0
30	0.0217	0.0466	0.0453	0.4783	0.0428	0.0439	0.0073	0.0406
45	0.0644	0.1517	0.1427	0.6427	0.1250	0.1328	0.0515	0.1100
60	0.1259	0.3424	0.3071	0.7401	0.2392	0.2649	0.1932	0.1897
75	0.18064	0.6271	0.5295	0.7795	0.3459	0.4019	0.5055	0.2420
90	0.2146	1.0000	0.7854	0.7854	0.3927	0.4674	1.0387	0.2530
105	0.1746	1.4402	1.0413	0.7913	0.3343	0.3635	1.7799	0.2718
120	0.0353	1.9132	1.2637	0.8307	0.1536	−0.0277	2.7395	0.5330
135	−0.2210	2.3732	1.4281	0.9281	−0.1250	−0.8208	3.4430	0.9170
150	−0.5925	2.7673	1.5255	1.0925	−0.4355	−2.1077	4.0390	1.9423

Plate 3.1. Helical staircase in concrete with wood/steel balustrade (with compliments from London Hilton Public Relations Department).

Helical part

$$
\left.
\begin{aligned}
v_y &= q_L r \overline{X}_5 \cos\phi && \text{(a)}\\
v_z &= -q_L r\left[\cos\alpha(\phi) + \overline{X}_5 \sin\alpha \sin\phi\right] && \text{(b)}\\
n &= -q_L r\left[\sin\alpha\phi - \overline{X}_5 \cos\alpha \sin\alpha\right] && \text{(c)}\\
m_y &= -q_L r^2\big[ke(1-\cos\phi) \\
&\qquad + \overline{X}_5 \tan\alpha\phi \sin\phi - \overline{X}_6 \cos\phi\big] && \text{(d)}\\
m_z &= q_L r^2\big[\sin\alpha(\phi - ke\sin\phi) \\
&\qquad - \overline{X}_5 \cos\alpha(\sin\phi + \tan^2\alpha\phi\cos\phi) \\
&\qquad - \overline{X}_6 \sin\alpha\sin\phi\big] && \text{(e)}\\
m_t &= -q_L r^2\big[\cos\alpha(\phi - ke\sin\phi) \\
&\qquad + \overline{X}_5 \sin\alpha(\sin\phi - \phi\cos\phi) \\
&\qquad - \overline{X}_6 \cos\alpha\sin\phi\big] && \text{(f)}
\end{aligned}
\right\} \quad (3.125)
$$

For the lengths of the length L of the horseshoe

$$
\left.\begin{aligned}
&v_y = q_L r \overline{X}_5 \cos\phi_0 && \text{(a)}\\
&v_z = \mp q_L r\left[\cos\alpha(\phi_0+\overline{\eta}) + \overline{X}_5 - \sin\alpha\sin\phi_0\right] && \text{(b)}\\
&n = \mp q_L r\left[\sin\alpha(\phi_0+\overline{\eta}) - \overline{X}_5 - \cos\alpha\sin\phi_0\right] && \text{(c)}\\
&m_y = -q_L r^2\Big[ke(1-\cos\phi_0) + \phi_0\overline{\eta} + \frac{1}{2}(\overline{\eta})^2\\
&\qquad + \overline{X}_5\tan\alpha\sin\phi_0(\phi_0+\overline{\eta}\cos\alpha) - \overline{X}_6\cos\phi_0\Big] && \text{(d)}\\
&m_z = \pm q_L r^2\Big[\sin\alpha(\phi_0 - ke\sin\phi_0) - \overline{X}_6\sin\alpha\sin\phi_0\\
&\qquad - \overline{X}_5\cos\alpha\Big(\sin\phi_0 + \tan^2\alpha\phi_0\cos\phi_0\\
&\qquad\qquad + \frac{\cos\phi_0}{\cos^2\alpha}\overline{\eta}\Big)\Big] && \text{(e)}\\
&m_t = \mp q_L r^2\left[\cos\alpha(\phi_0 - ke\sin\phi_0)\right.\\
&\qquad + \overline{X}_5\sin\alpha(\sin\phi_0 - \phi_0\cos\phi_0)\\
&\qquad \left. - \overline{X}_6\cos\alpha\sin\phi_0\right] && \text{(f)}
\end{aligned}\right\} \quad (3.126)
$$

Note that for $\overline{X} = \delta_{50}/q_L r^2$ etc. and $\overline{\eta} = x/L$

$\overline{X} = \delta_{60}/q_L r^2$

For the interaction of the straight and curved parts, the modified values of f^* are written as:

$$
\begin{aligned}
f_{55}^* &= f_{55} + 2\theta_y\tan^2\alpha\sin^2\phi_0(\phi_0+\rho\cos\alpha)^2\\
&\quad + 2\theta_x(\sin\phi_0+\rho\cos\phi_0)^2\\
&\quad + 2\theta_x\tan^2\alpha\cos^2\phi_0(\phi_0+\rho)^2\\
f_{56}^* &= f_{56}\theta_y\tan\alpha\sin 2\phi_0(\phi_0+\rho\cos\alpha)\\
&\quad + \theta_x\tan\alpha\sin 2\phi_0(\phi_0+\rho)\\
f_{66}^* &= f_{66} + 2\theta_y\cos^2\phi_0 + 2\theta_x\sin^2\phi\\
\delta_{50}^* &= \delta_{50} + 2\theta_y q_L r^2\tan\alpha\sin\phi_0(\phi_0+\rho\cos\alpha)\\
&\qquad \times\left[ke(1-\cos\phi_0) + \rho\phi_0 + \frac{\rho^2}{2}\right]\\
&\quad - 2\theta_x q_L r^2\tan\alpha\cos\phi_0(\phi_0+\rho)(\phi_0 - ke\sin\phi_0)\\
\delta_{60}^* &= \delta_{60} - 2\theta_y q_L r^2\cos\phi_0\\
&\qquad \times\left[ke(1-\cos\phi_0) + \rho\phi_0 + \frac{\rho^2}{2}\right]\\
&\quad - 2\theta_x q_L r^2\sin\phi_0(\phi_0 - ke\sin\phi_0)
\end{aligned}
\qquad (3.127)
$$

for the purpose of tabulation

$$\delta_{50}^{*1} = \frac{\delta_{50}^{*}}{q_L r^2}, \quad \delta_{60}^{*1} = \frac{\delta_{60}^{*}}{q_L r^2} \tag{3.127a}$$

EXAMPLE 3.13: *Analysis of a horseshoe helical staircase*

A helical type horseshoe staircase is shown in Figure 3.35.

Using the following data, analyse this staircase for displacements, redundants, shear, moments and axial thrusts.

Data

$H_1 = 4.25$ m

$2n = 26$, $n = 13$ one side of the horseshoe

$h_1 = 0.1635$

$\overline{G} = 0.3$ m, $g_k = 5.72$ kN/m^2

$B = 1.40$ m, $q_k = 3.125$ kN/m^2

$r_i = 0.5$ m

$r = 1.20$ m

$\phi_0 = 105°$

$L_t = 2n\overline{G} = 7.8$ m

$e = B^2/12r$

$L = \dfrac{L_t - L_{2PO}}{2} = 1.7$ m

D_f = stair slab thickness 0.16 m or 160 mm

SOLUTION

$L_t = 2n\overline{G} = 26 \times 0.3 = 7.8$ m

$w = 1.4g_k + 1.69q_k = 1.4 \times 5.72 + 1.6 \times 3.125 = 13$ kN/m^2

L_{2PO} = Sector $AC = 15 \times 0.3 = 4.5$

From sector of circle with 210° angle 4.45 (average $L_{2PO} = 4.4$)

$$L = \frac{L_t - L_{2PO}}{2} = \frac{7.8 - 4.4}{2} = 1.7 \text{ m}$$

$$\beta = \frac{I_y}{I_z} = \left(\frac{D_f}{B}\right)^2 = 0.0131$$

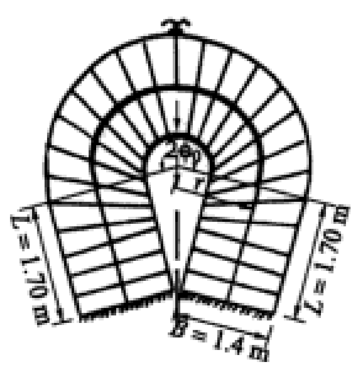


Figure 3.35. A helical type horseshoe staircase.

$$ke = 1 + \frac{e}{r} = 1 + \frac{B^2}{12r^2} = 1.1134$$

$$= \frac{L}{r} = 1.417$$

$a_1 = 0.1745, \quad a_2 = 1.4401, \quad a_3 = 1.0412, \quad a_4 = 0.7912$

$a_5 = 0.3343, \quad a_6 = 0.3635, \quad a_7 = 1.7800, \quad a_8 = 0.2718$

$f_{55} = 3.486, \quad f_{56} = -0.279, \quad f_{66} = 1.809, \quad \delta_{50} = 6.89 q_L r^2$

$\delta_{60} = 0.404 q_L r^2$

Looking at the moment for a single span staircase (M_y)

$f^*_{55} = 3.486 + 5.246(\theta_y), \qquad f^*_{56} = -0.279 + 0.838(\theta_y)$

$f^*_{66} = 1.809 + 0.134(\theta_y), \qquad \delta^*_{50} = 6.898 + 16.203(\theta_y) q_L r^2$

$\delta^*_{60} = 0.404 + 2.589(\theta_y) q_L r^2$

for $q_L r^2 = 1$

$$\overline{X}_5 = \frac{-12.591 - 30.618\theta_y}{6.228 + 10.425}, \quad \overline{X}_6 = \frac{-3.333 - 9.884\theta_y}{6.228 + 10.425\theta_y}$$

for

$\theta_y = 0, \quad \overline{X}_5 = -2.022$ and $\overline{X}_6 = -0.535$

$\theta_y = \frac{1}{2}, \quad \overline{X}_5 = -2.439$ and $\overline{X}_6 = -0.723$

$\theta_y = \infty, \quad \overline{X}_5 = -2.937$ and $\overline{X}_6 = -0.948$

3.6 STIFFNESS METHOD

In this method the unknown involved is the displacement of the joints of a particular staircase. There are essentially two principal ways in which the equations of equilibrium, kinetmatics and elasticity can be combined to lead to a set of equations in displacements. The two approaches are: (a) the basic stiffness method by involving basic stiffness matrices of members and (b) the direct stiffness method by involving the general stiffness of members.

The common objective is finally to obtain the following set of equations:

$$\{P\} = [K]\{\delta\} \tag{3.128}$$

where $\{P\}$ are joint forces, $\{\delta\}$ is the corresponding displacement and $[K]$ is the stiffness matrix.

3.6.1 *Basic stiffness method*

The Eq. (3.128) can be expressed as equations of equilibrium in terms of joint displacements.

$$\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ \vdots & & & & \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{Bmatrix} \tag{3.129}$$

The coefficients k_{11} etc are obtained in terms of the basic stiffness $[K]$ of members. The deformation of members is related to the displacement of the joints of the staircase(s) members. Having solved for displacements, it becomes easier to compute the internal forces for each member of component of the staircase.

Member deformation

The deformed shape of a member can be evaluated by (b) the beam model

(i) In the beam model, the rigid body motion under consideration (Fig. 3.35) involves a rotation $v_2 - v_1/L$ and a translation u_1 and u_2. The member deformations are obtained by subtracting the rigid body effects.

$$\text{Chord rotation} \left\langle \begin{matrix} \phi_1 = \theta_1 - \dfrac{v_2 - v_1}{L} \\ \phi_2 = \theta_2 - \dfrac{v_2 - v_1}{L} \end{matrix} \right\rangle \begin{matrix} \text{rigid body} \\ \text{displacements} \end{matrix} \tag{3.130}$$

axial elongation $e = u_2 - u_1$

(ii) In the cantilever model, the rigid body displacements of a staircase member (Fig. 3.36) involves a translation u_1, u_2 and a rotation θ_1. The deformation relative to end '1' being fixed can be expressed as:

$$v = v_2 - v_1 - L\theta_1 \text{ at end 2 in a tranverse direction} \tag{3.131}$$

$$\phi = \theta_2 - \theta_1$$

$$e = u_2 - u_1$$

These two models correspond exactly to the member forces which are defined as a set of independent forces for a member. The cantilever model is relatively easier. The beam displacements at end 2 can be obtained

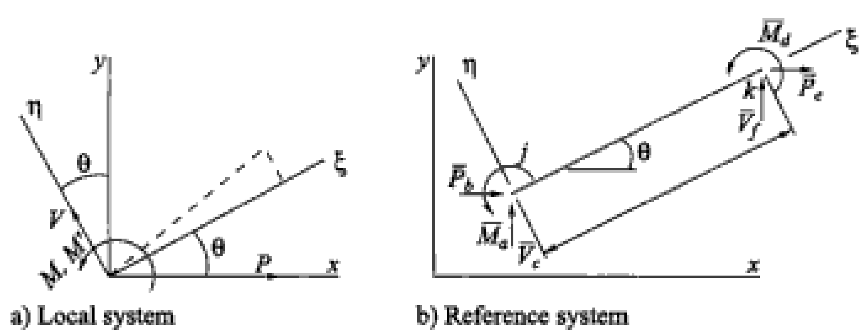


Figure 3.36. Reference coordinate systems.

when the proportion of x of the beam is assumed to act as a rigid body.

$$\begin{aligned} de &= du \\ dv &= x\,d\theta = d\theta = d\phi \end{aligned} \tag{3.132}$$

Total deformation is obtained by summing over the entire beam.

$$e = \int_0^L du, \quad v = \int_0^L x\,d\theta, \quad \phi = \int_0^L d\theta \tag{3.133}$$

$$\phi_1 = -\frac{v}{L} = -\int_0^L \frac{x}{L}\,d\theta, \quad \phi_2 = \phi - \frac{v}{L} = \int_0^L \left(1 - \frac{x}{L}\right) d\theta \tag{3.134}$$

$$-\frac{v}{L} = \text{rotation at end 1} \tag{3.134a}$$

$$\phi - \frac{v}{L} = \text{rotation at end 2} \tag{3.134b}$$

Typical stiffness coefficients and standard cases are given in Table 3.16.

3.6.2 *Direct stiffness method*

The slope deflection equation is in fact a direct stiffness method. First the general stiffness matrices of various members are expressed in global coordinates. The equations of equilibrium are written between internal member forces in global coordinates and joint loads. A typical example is given explaining various principles. The general stiffness matrix can be expressed in terms of loads and displacements as:

$$\underset{p}{\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}} = \underset{k_{ij}}{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}} \underset{\delta}{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} \tag{3.135}$$

The k_{ij} matrices (i, $j = 1, 2$) can be obtained in terms of kinematic matrices of the member and the coordinate transformation matrix.

$$\begin{aligned} k_{ij} &= T_e \bar{k}_{ij} T_e^T = A_i^T k A_j \\ k_{ij} &= \overline{A}_i k A_j \end{aligned} \tag{3.136}$$

The general 'k_{ij}' matrix is given in Table 3.17.

3.6.3 *Transformation technique*

End actions and end displacements are generally defined with respect to the local axes of the element (ξ, η) in two dimensions or (ξ, η, ζ) in three dimensions. If they are referred back to the global axes (x, y) or (x, y, z) the relationship can easily be established between them and the local axes. Assuming transformation is required for a two dimensional case, in which Figure 3.36 shows axial forces P, shears V and bending moments M on referred global axes x, y then the local values $\overline{P}, \overline{V}$ and

Table 3.16. Stiffness coefficients – standard cases.

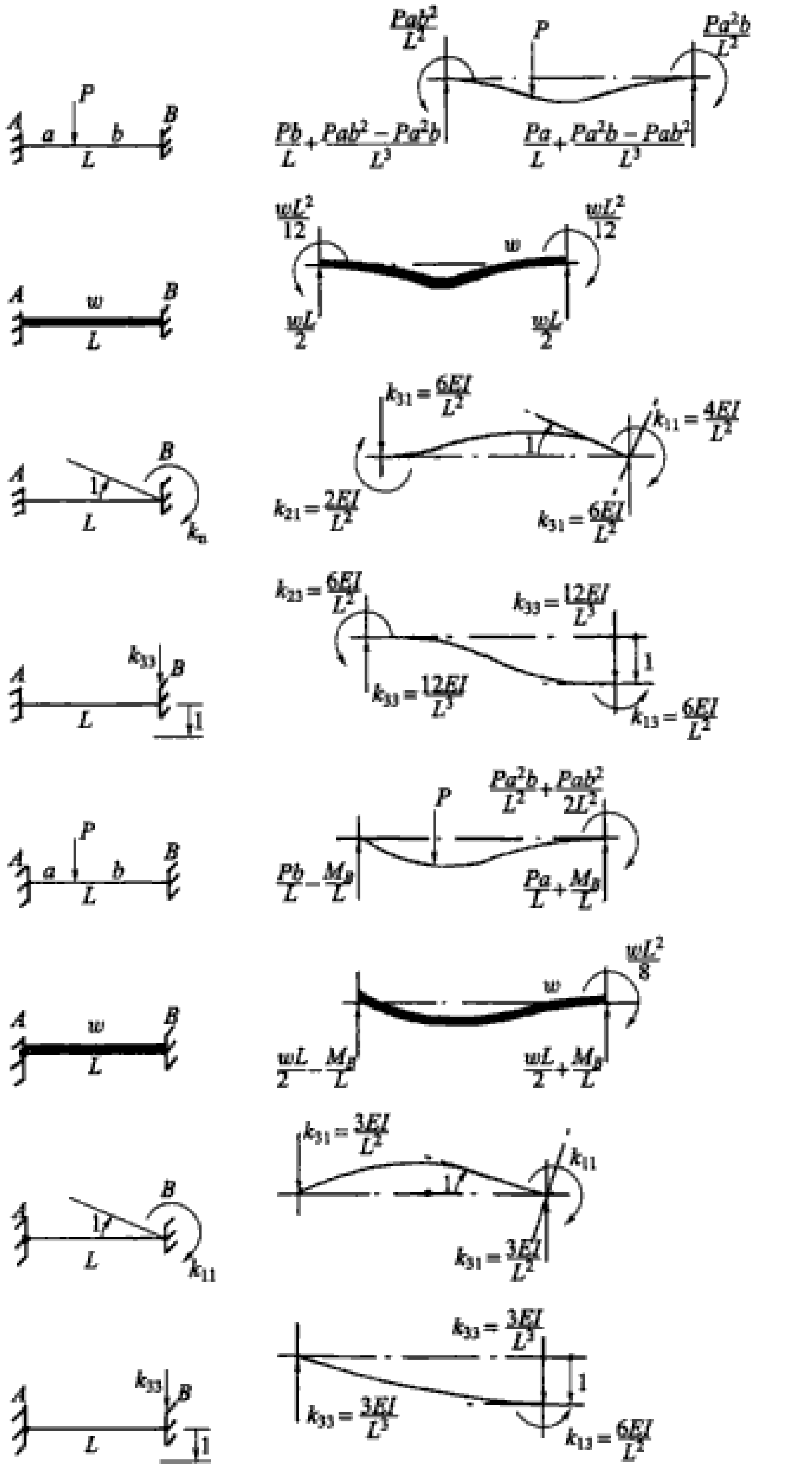


Table 3.17. k_{ij} matrices – plane frame member.

$k_5c^2 + k_3s^2$	$(k_5 - k_3)cs$	$-k_4s$	$-(k_5c^2 + k_3s^2)$	$-(k_5 - k_3)cs$	$-k_4s$
$(k_5 - k_3)cs$	$k_3c^2 + k_5s^2$	k_4c	$-(k_5 - k_3)cs$	$-(k_3c^2 + k_5s^2)$	k_4c
$-k_4s$	k_4c	k_1	k_4s	$-k_4c$	k_2
$-(k_5c^2 + k_3s^2)$	$-(k_5 - k_3)cs$	k_4	$k_5c^2 + k_3s^2$	$(k_5 - k_3)cs$	k_4s
$-(k_5 - k_3)cs$	$-(k_3c^2 + k_5s^2)$	$-k_4c$	$(k_5 - k_3)cs$	$k_3c^2 + k_5s^2$	$-k_4c$
$-k_4s$	$-k_4c$	k_2	k_4s	$-k_4c$	k_1

$k_1 = 4EI/L$; $k_2 = 2EI/L$; $k_3 = 12EI/L^3$; $k_4 = 6EI/L^2$; $k_5 = AE/L$ $c = \cos\alpha$; $s = \sin\alpha$

$\overline{M}$ can be related as:

$$\overline{P} = p\cos\theta - v\sin\theta \tag{3.137}$$

$$\overline{M} = m \tag{3.138}$$

$$\overline{V} = p\sin\theta + v\cos\theta \tag{3.139}$$

In a matrix form

$$\begin{Bmatrix} \overline{M} \\ \overline{P} \\ \overline{V} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} p \\ m \\ v \end{Bmatrix} \tag{3.140}$$

Considering a typical j position in the frame (Fig. 3.36) of an element, the following relationship can be established

$$\begin{Bmatrix} \overline{M}_a \\ \overline{P}_b \\ \overline{V}_c \\ \overline{M}_d \\ \overline{P}_e \\ \overline{V}_f \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} m_a \\ p_b \\ v_c \\ m_d \\ p_e \\ v_f \end{Bmatrix} \tag{3.141}$$

for the element 'i'

$$\{\overline{M}_e\}_i = [T_e]_i\{m_e\}_i \tag{3.142}$$

where

$$[T_e] = \begin{bmatrix} [t] & [0] \\ [0] & [t] \end{bmatrix}_i \tag{3.143}$$

Equation (3.145) is also the end transformation matrix

$$[T_e] = [t]_1 \tag{3.144}$$

The general displacement of the element is written as:

$$\{\overline{\delta}_e\}_i = [T_e]_i\{\delta_e\}_i \tag{3.145}$$

The inverse of the element transformation matrix is equal to the transpose of the matrix

$$\{\delta_e\}_i = [T_e]^T\{\overline{\delta}_e\}_i \tag{3.146}$$

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3.6.5 *Stiffness values and reactions*

Prior to the evaluation of unit values-reactions, it is important to know the values of direction cosines c and s for this staircase.

$$\text{Element 1 } c = \frac{L_1}{\sqrt{(L_1^2 + h^2)}}, \quad s = \frac{h}{\sqrt{(L_1^2 + h^2)}}$$
$$\text{Element 2 } c = \frac{L_2}{L_2} = 1, \quad s = 0 \tag{3.158}$$

Various deformed diagrams for the reactions for various displacements and rotations are shown in Figure 3.42 for restrained flights of staircases.

3.6.6 *The stiffness matrix*

$[k_{TOT}]_1$ = total stiffness matrix for element (1)

displacement at k-end actions at j-end

$$= \left[\begin{array}{ccc:ccc} \bar{k}_{77} & \bar{k}_{78} & \bar{k}_{79} & \bar{k}_{71} & \bar{k}_{72} & \bar{k}_{73} \\ \bar{k}_{87} & \bar{k}_{88} & \bar{k}_{89} & \bar{k}_{81} & \bar{k}_{82} & \bar{k}_{83} \\ \bar{k}_{97} & \bar{k}_{98} & \bar{k}_{99} & \bar{k}_{91} & \bar{k}_{92} & \bar{k}_{93} \\ \hdashline \bar{k}_{17} & \bar{k}_{18} & \bar{k}_{19} & \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{27} & \bar{k}_{28} & \bar{k}_{29} & \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} \\ \bar{k}_{37} & \bar{k}_{38} & \bar{k}_{39} & \bar{k}_{31} & \bar{k}_{32} & \bar{k}_{33} \end{array}\right] \tag{3.159}$$

displacement at j-end actions at k-end

Similarly the components of the transformed element stiffness matrix can be identified as the following for the elements (2)

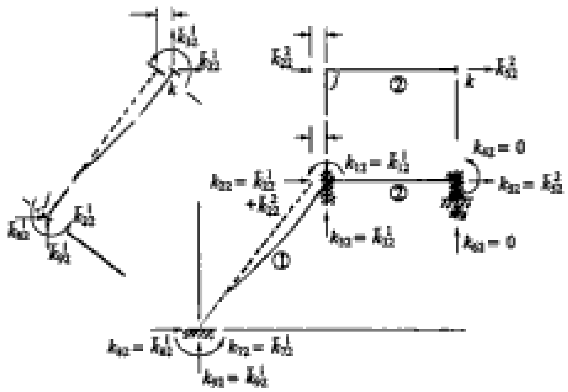
action at j-end

$$[k_{TOT}]_2 = \left[\begin{array}{ccc:ccc} \bar{k}_{11} & \bar{k}_{12} = 0 & \bar{k}_{13} & \bar{k}_{14} & \bar{k}_{15} = 0 & \bar{k}_{16} \\ 0 & \bar{k}_{22} & 0 & 0 & \bar{k}_{25} & 0 \\ \bar{k}_{13} & 0 & \bar{k}_{33} & \bar{k}_{34} & 0 & \bar{k}_{36} \\ \hdashline \bar{k}_{41} & 0 & \bar{k}_{43} & \bar{k}_{44} & 0 & \bar{k}_{46} \\ 0 & \bar{k}_{52} & 0 & 0 & \bar{k}_{55} & 0 \\ \bar{k}_{61} & 0 & \bar{k}_{63} & \bar{k}_{64} & 0 & \bar{k}_{66} \end{array}\right] \tag{3.160}$$

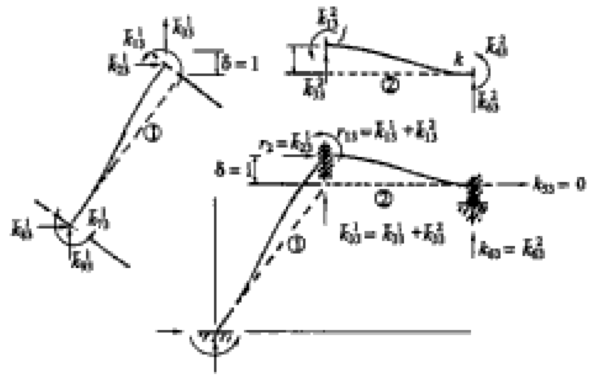
displacement j-end displacement k-end

Tables 3.18 and 3.19 gives a complete picture of other types of deformation for the same restrained staircase while looking at appropriate unit displacement and rotations against corresponding reactions, respectively. The reactions can be summarised in Equations (3.161) and (3.162) while

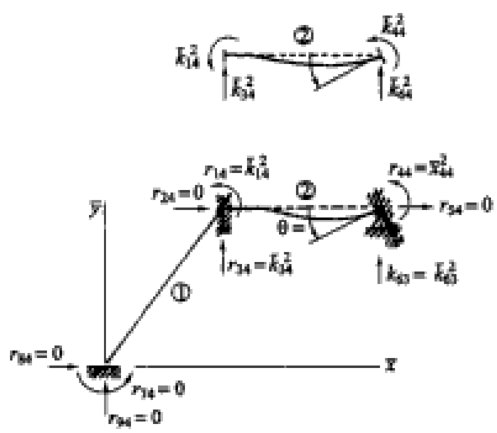
Table 3.18. Joint rotation and translation in $\overline{X}$ and $\overline{Y}$ directions joints 2-5.



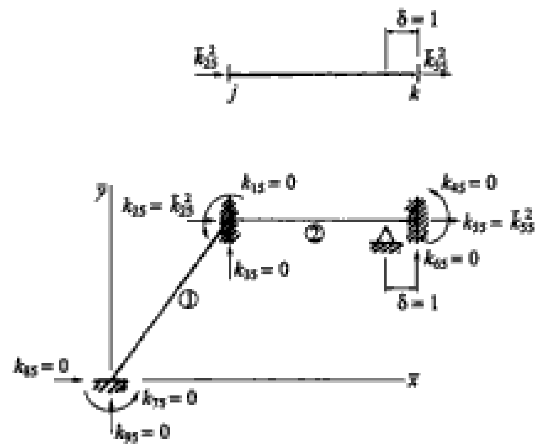
a) A unit displacement at 2 (translation in $\overline{X}$-direction)



b) A unit displacement at 4 (rotation of joint 3)



c) A unit displacement at 3 (translation in $\overline{Y}$-direction)

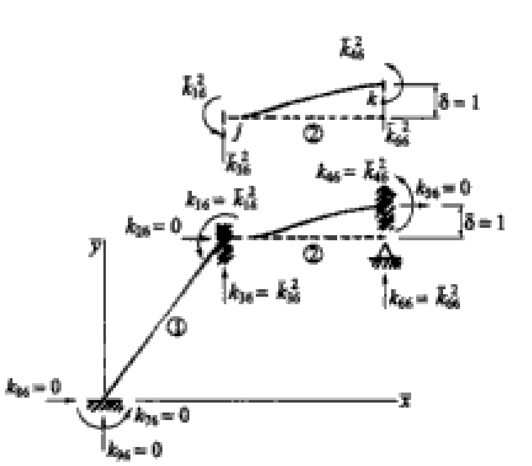


d) A unit displacement at 5 (translation of joint 3 in $\overline{X}$-direction)

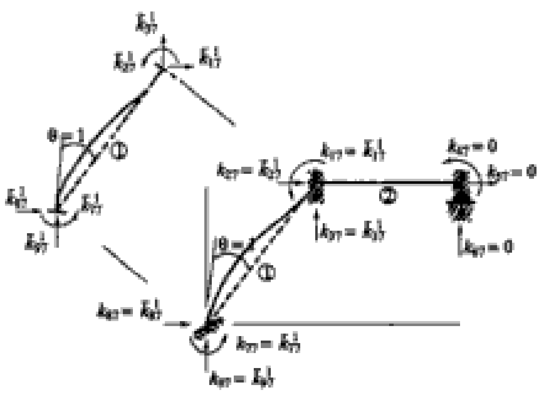
considering:

$$
\begin{aligned}
&k_{11} = \bar{k}^1_{11} + \bar{k}^2_{11}, && \bar{k}^1_{91} = k_{91} \\
&k_{21} = \bar{k}^1_{21}, && k_{36} = k_{63} = \bar{k}^2_{36} \\
&k_{31} = \bar{k}^1_{31} + \bar{k}^2_{31}, && k_{33} = \bar{k}^1_{33} + \bar{k}^2_{33} \\
&k_{41} = k_{14} = \bar{k}^2_{41}, && k_{34} = k_{43} = \bar{k}^2_{34} \\
&k_{51} = 0, && k_{44} = \bar{k}^2_{44} \\
&k_{61} = \bar{k}^2_{61}, && k_{25} = \bar{k}^2_{25} \\
&k_{71} = \bar{k}^1_{71}, && k_{66} = \bar{k}^2_{66} \\
&k_{81} = \bar{k}^1_{81}
\end{aligned}
\tag{3.161}
$$

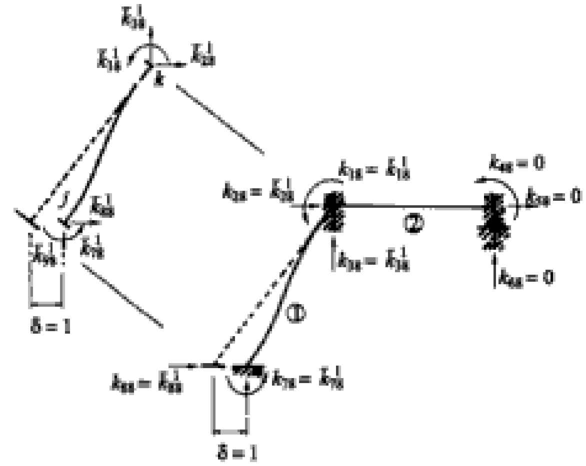
Table 3.19. Joint rotation and translation in X and Y directions joints 6-9.



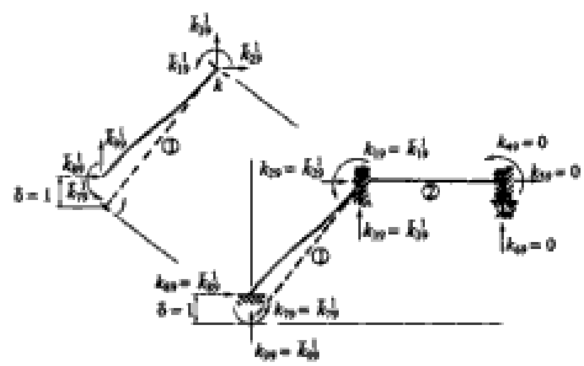
a) A unit displacement at 6 (translation of joint 3 in $\overline{Y}$-direction)



b) A unit displacement at 7 (rotation of joint 1)



c) A unit displacement at 8 (translation of joint 4)



d) A unit displacement at 9 (translation of joint 1 in $\overline{Y}$-direction)

$$
\begin{array}{cccc:ccccc}
 & [K_{uu}] & & & & [K_{ur}] & & & \\
k_{11} & k_{12} & k_{13} & k_{14} & 0 & k_{16} & k_{17} & k_{18} & k_{19} \\
k_{21} & k_{22} & k_{23} & 0 & k_{25} & 0 & k_{27} & k_{28} & k_{29} \\
k_{31} & k_{32} & k_{33} & k_{34} & 0 & k_{36} & k_{37} & k_{38} & k_{39} \\
k_{41} & 0 & k_{43} & k_{44} & 0 & k_{46} & 0 & 0 & 0 \\
\hdashline
0 & k_{52} & 0 & 0 & k_{55} & 0 & 0 & 0 & 0 \\
k_{61} & 0 & k_{63} & k_{64} & 0 & k_{66} & 0 & 0 & 0 \\
\{k_{71} & k_{72} & k_{73} & 0 & 0 & 0 & k_{77} & k_{78} & k_{79}\} \\
\{k_{81} & k_{82} & k_{83} & 0 & 0 & 0 & k_{87} & k_{88} & k_{89}\} \\
\{k_{91} & k_{92} & k_{93} & 0 & 0 & 0 & k_{97} & k_{98} & k_{99}\} \\
 & [K_{ru}] & & & & [K_{rr}] & & &
\end{array}
\qquad (3.162)
$$

$k_{61} = \bar{k}_{61}$
$k_{64} = \bar{k}^2_{64}$
$k_{55} = \bar{k}^2_{55}$

all other values marked by $*\{,\}*$ are $k = k^{-1}$ with appropiate subscript.

The two reactions at supports acting on an indeterminate stair flight must be checked.

For example,

$$R_{1\bar{R}} + k_{11}\delta_1 + k_{12}\delta_2 + \cdots + k_{19}\delta_{19} \tag{3.163}$$

$$R_{4\bar{R}} + k_4\delta_4 = 0 \tag{3.164}$$

$$R_{5\bar{R}} + k_5\delta_5 = R_5 \tag{3.165}$$

right up to $R_{9\bar{R}} + k_{91}\delta_1 + \cdots + k_{99}\delta_9 = R_9$ (3.166)

only the value $R_{7\bar{R}} = M_7$

$$\left.\begin{array}{l} R_8 = \text{horizontal force } H \\ R_9 = \text{vertical reaction } v_9 \\ R_{7\bar{R}} = M_7 = \text{moment} \end{array}\right\} \text{at nodal point (1)}$$

$R_5 = H =$ at nodal point (3) $= H_5$
$R_6 =$ vertical reaction at nodal point (3) $= v_6$

A reference is made to Figure 3.43 for various values of R, for the support reactions.

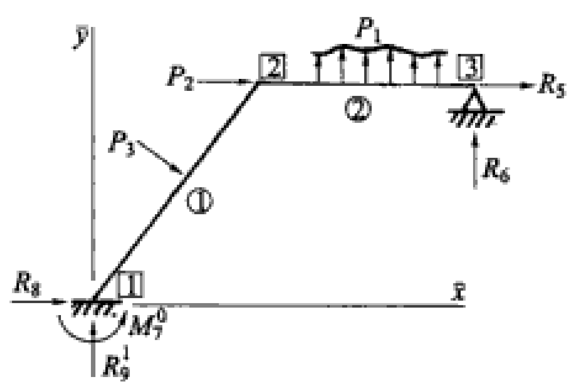


Figure 3.43. Identification of components of support reaction for original structure of staircase.

Support reactions

The following gives the complete picture of various actions on the staircase.

$$\begin{Bmatrix} M^{F1}+M_1^{F2} \\ P_2^{F1}+P_2^{F2}-P_2 \\ V_3^{F1}+V_3^{F2} \\ M_4^{F2} \\ 0 \\ V_7^{F2} \\ M_7^{F1} \\ P_8^{F1} \\ V_9^{F1} \end{Bmatrix} + \left[\begin{array}{c:c} K_{UU} & K_{UR} \\ \hdashline K_{RU} & K_{RR} \end{array}\right] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ H_5 \\ V_6 \\ M_7 \\ V_8 \\ H_8 \end{Bmatrix} \tag{3.167}$$

In general Eq. (3.167) is written as

$[K]_{TOT}\{\delta\}_{TOT} = \{P\}^J + \{R\}_{TOT}$

$[K]_{TOT}$ = total structure stiffness matrix

$\{\delta\}_{TOT}$ = total joint displacement matrix

$\{P\}^J$ = total joint load matrix corresponding to P_u unrestrained and P_r restrained of joint displacement, respectively

$\{R\}_{TOT}$ = total support reaction matrix of the original structure corresponding to the restrained component of joint displacement

u, r = unrestrained and restrained

f = final

Equation (3.150) can also be written as

$$\left[\begin{array}{c:c} K_{uu} & K_{ur} \\ \hdashline K_{ru} & K_{rr} \end{array}\right] \left\{\begin{array}{c} \delta_u \\ \hdashline \delta_r \end{array}\right\} = \left\{\begin{array}{c} P_u \\ \hdashline P_r \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \hdashline R_r \end{array}\right\}$$

Using Eq. (3.147) for element (1)

$$\begin{Bmatrix} M_7^f \\ P_8^f \\ V_9^f \\ M_1^f \\ P_2^f \\ V_3^f \end{Bmatrix} = [K]_{TOT_1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} + \begin{Bmatrix} M_7^F \\ P_8^F \\ V_9^F \\ M_1^F \\ P_2^F \\ V_3^F \end{Bmatrix}$$

for element (2)

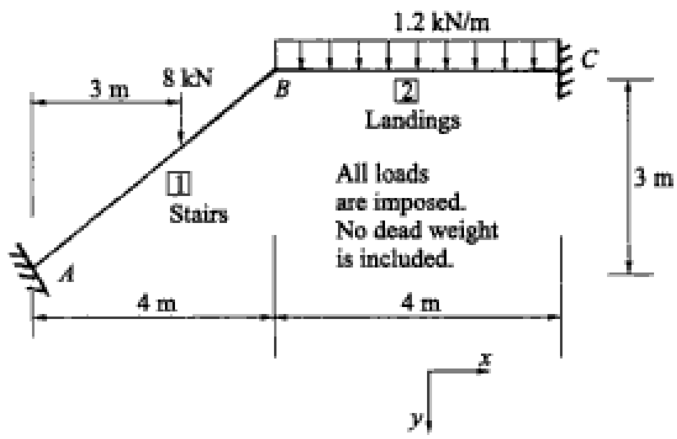
$$\begin{Bmatrix} M_1^f \\ P_2^f \\ V_3^f \\ M_4^f \\ P_5^f \\ V_6^f \end{Bmatrix} = [K]_{TOT_2} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} M_1^F \\ P_2^F \\ V_3^F \\ M_4^F \\ P_5^F \\ V_6^F \end{Bmatrix}$$

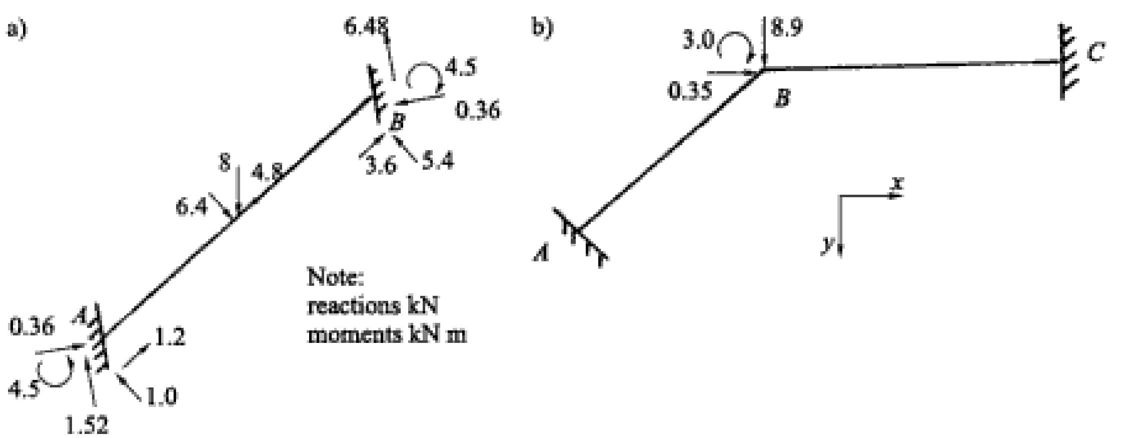
EXAMPLE 3.14: *A staircase analysis using stiffness method*
A staircase shown in Figure 3.44 is fixed at A and C. The landing and the flight are loaded as shown. Assume that all the loads on the steps, together with the weight of the stairs, concentrate at the centroid. Use the following member properties:

M	L (m)	$\cos\alpha$	$\sin\alpha$	EI/EI_0	r	KI/EI_0	$K2/EI_0$	$K3/EI_0$	$K4/EI_0$	$K5/EI_0$
AB (1)	5	0.8	−0.6	12.5	20	10	5	1.2	3	40
BC (2)	4	1	0	8	20	8	4	1.5	3	50

$EI_{(1)}/EI_{(2)} = 12.5/8$ etc.

Figure 3.44. The resistance will be opposite sign.





SOLUTION

Reactions:
When all points A, B and C are fixed. From statics the load in members is given in Figure 3.44(a) and (b).

$$R_c \uparrow = -2.4 \text{ kN} \qquad R_B \uparrow = -8.9 \text{ kN}$$
$$H_B = -0.35 \text{ kN} \qquad M_c = 1.6 \text{ kN m}$$
$$M_B = 3.0 \text{ kN m} \qquad R_A \uparrow = 1.52 \text{ kN}$$
$$H_A = 0.35 \text{ kN} \rightarrow \qquad M_A = -1.5 \text{ kN m}$$

k_{ij} matrices (ref. Table 3.17)

$$k_5c^2 + k_3s^2 = 40(0.64) + 1.2(0.36) = 26.032$$
$$k_5s^2 + k_3c^2 = 15.168$$
$$(k_5 - k_3)cs = -18.624$$
$$k_4c = 3.0 \times 0.8 = 2.4$$
$$k_4s = -1.8$$

k_{ij} is written as

$$[k_{ij}]_{(1)} = \left[\begin{array}{ccc:ccc} 26.032 & -18.624 & 1.8 & -26.032 & 18.624 & 1.8 \\ -18.624 & 15.168 & 2.4 & 18.624 & -15.168 & 2.4 \\ 1.8 & 2.4 & 10.0 & -1.8 & -2.4 & 5.0 \\ \hdashline -26.032 & 18.624 & -1.8 & 26.032 & -18.624 & -1.8 \\ 18.624 & -15.168 & -2.4 & -18.624 & 15.168 & -2.4 \\ 1.8 & 2.4 & 5.0 & -1.8 & -2.4 & 10.0 \end{array}\right]$$

$[k_{ij}]$ for A and B are then written as

$$[k_{ij}]_{(1)} = \begin{bmatrix} 50.0 & 0 & 0 & -50.0 & 0 & 0 \\ 0 & 1.5 & 3.0 & 0 & -1.5 & 3.0 \\ 0 & 3.0 & 8.0 & 0 & -3.0 & 4.0 \\ -50.0 & 0 & 0 & 50 & 0 & 0 \\ 0 & -1.5 & -3.0 & 0 & 1.5 & -3.0 \\ 0 & 3.0 & 4.0 & 0 & -3.0 & 8.0 \end{bmatrix}$$

Adding $k_{22_{(1)}}$ and $k_{11_{(2)}}$ and solving for $\{\delta\}$

$$\{\delta\} = \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix} = \frac{1}{EI_0}\begin{Bmatrix} 0.183 \\ 0.743 \\ -0.170 \end{Bmatrix}$$

The member forces in stair and in landing

$$\begin{Bmatrix} R_A \\ V_{1A} \\ M_{1A} \end{Bmatrix} = [K_{\text{type}}]\begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix}$$

$$= \begin{bmatrix} -26.032 & 18.624 & 1.8 \\ 18.624 & -15.168 & 2.4 \\ -1.8 & -2.4 & 5.0 \end{bmatrix}\begin{Bmatrix} 0.183 \\ 0.743 \\ -0.170 \end{Bmatrix} = \begin{Bmatrix} 8.80 \\ -8.30 \\ -3.00 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN m} \end{matrix}$$

The other member forces are computed as:

$$\begin{Bmatrix} P_{2A} \\ V_{2A} \\ M_{2A} \end{Bmatrix} = \begin{Bmatrix} -8.80 \\ 8.30 \\ -3.80 \end{Bmatrix} \text{kN}$$

$$\begin{Bmatrix} P_{1B} \\ V_{1B} \\ M_{1B} \end{Bmatrix} = \begin{Bmatrix} 9.150 \\ 0.610 \\ 0.900 \end{Bmatrix} \text{kN}$$

$$\begin{Bmatrix} P_{2B} \\ V_{2B} \\ M_{2B} \end{Bmatrix} = \begin{Bmatrix} -9.140 \\ -0.610 \\ 1.560 \end{Bmatrix} \text{kN}$$

Superimposing all these

$$\begin{Bmatrix} P_{1A} \\ V_{1A} \\ M_{1A} \end{Bmatrix} = \begin{Bmatrix} 9.130 \\ -9.800 \\ -4.450 \end{Bmatrix} \text{kN}$$

$$\{P_{2A}\}^T = [-9.140\ 1.800\ 0.710]\ \text{kN}$$

$$\{P_{1B}\}^T = [9.140\ 1.800\ -0.710]\ \text{kN}$$

$$\{P_{2B}\}^T = [-9.150\ -3.010\ 3.160]$$

3.7 FINITE ELEMENT METHOD

3.7.1 *Introduction*

The two types of finite element analysis are: plate flexure analysis using displacement polynomials and isoparametric finite element analysis.

Both methods are useful. If the reader is interested in evaluating moments, shear and axial effects in two dimensions, the plate flexure method is adopted. Where the reader wishes to carry out an in depth study, the isoparametric finite element method is adopted which, apart from stresses, strains, yielding and plasticity, also looks at cracks in three directions and their propagation under ultimate loads.

3.7.2 *Plate flexure analysis using finite element*

The coordinates and the node numbering system can be defined for the rectangular element. They are given in Figure 3.45. The dimensions of the plate are a, b and t (thickness) and the coordinates are (x, y, z) in the cartesian coordinate system. The nodes 1 to 4 have their respective rotations

θ_{x1} to θ_{x4}; nodal forces (F_{x1}, F_{y1}) to (F_{x4}, F_{y4}) and displacements $\overline{\omega}$

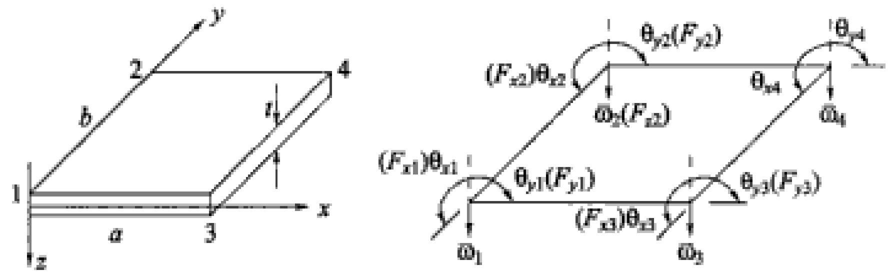


Figure 3.45. The rectangular finite element for plate flex.

In mathematical terms they are given as:

$$\begin{aligned}\overline{\omega} = \text{displacement} &= a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \\ &\quad + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \end{aligned} \tag{3.168}$$

$$\theta_x = -\frac{\partial \overline{\omega}^2}{\partial y}, \quad \theta_y = \frac{\partial \overline{\omega}}{\partial x} \qquad \text{(a) (3.169)}$$

$$\partial_1 = \left(\theta_{x1}, \theta_{y1}, \overline{\omega}_1\right) \qquad \text{(b)}$$

$$\{F_1\} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{Bmatrix} \tag{3.170}$$

The nodal loads are related to displacements as:

$$\{F\} = [K]\{\delta\} \tag{3.171}$$

where

$$\{F\} = \left[F_{x1}, F_{y1}, F_{z1}, \ldots, F_{x4}, F_{y4}, F_{z4}\right]^T \tag{3.172}$$

$$\{\delta\} = \left[\theta_{x1}, \theta_{y1}, w_{z1}, \ldots, \theta_{x4}, \theta_{y4}, \theta_{z4}\right]^T$$

where $[K]$ is the stiffness matrix, elemental or global.

Now

$$\begin{aligned}\theta_x = -\frac{\partial \overline{\omega}}{\partial y} &= -\big(a_3 + a_5x + 2a_6y + a_8x^2 + 2a_9xy + 3a_{10}y^2 \\ &\quad + a_{11}x^3 + 3a_{12}xy^2\big)\end{aligned} \tag{3.173}$$

$$\begin{aligned}\theta_y = -\frac{\partial \overline{\omega}}{\partial x} &= a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 \\ &\quad + 3a_{11}x^3y + a_{12}y^3\end{aligned} \tag{3.174}$$

for the edge 1-2 x = constant and equal to zero

$$\overline{\omega} = a_1 + a_3y + a_6y^2 + a_{10}y^3 \tag{3.175}$$

$$\theta_x = -\left(a_3 + 2a_6y + 3a_{10}y^2\right) \tag{3.176}$$

$$\theta_y = a_2 + a_5y + a_9y^2 + a_{12}y^3 \tag{3.177}$$

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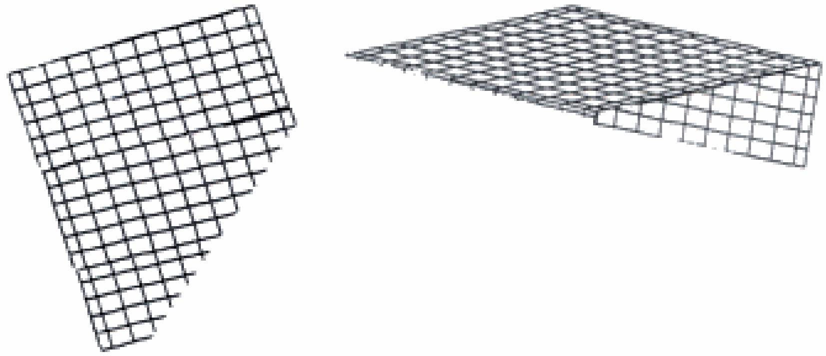

Figure 3.46. Typical mesh schemes for a flight or landing.

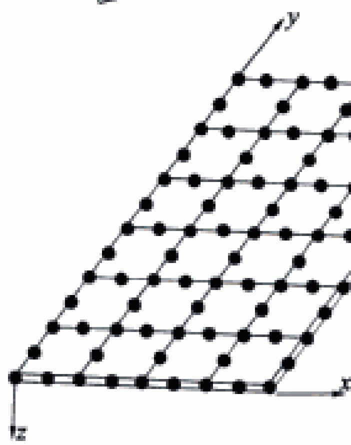

Figure 3.47. Typical landing or flight mesh scheme.

Table 3.20 (cont.).

$A = 20a^2 D_y + 8b^2 D_{xy}$	$P = 10a^2 D_y - 8b^2 D_{xy}$
$B = 15abD_1$	$R = -15\frac{a^2}{b} D_y + 15bD_1 + 6bD_{xy}$
$C = 20b^2 D_x + 8a^2 D_{xy}$	
$D = 30\frac{a^2}{b} D_y + 15bD_1 + 6bD_{xy}$	$G' = 5a^2 D_y + 2b^2 D_{xy}$
	$M' = 15\frac{a^2}{b} D_y - 6bD_{xy}$
$E = 30\frac{b^2}{a} D_x + 15aD_1 + 6aD_{xy}$	
	$T = 10b^2 D_x - 2a^2 D_{xy}$
$F = 60\frac{b^2}{a^2} D_x + 60\frac{b^2}{a^2} D_y + 30D_1 + 84D_{xy}$	$Y = 30\frac{b^2}{a} D_x + 6aD_{xy}$
$G = 10a^2 D_y - 2b^2 D_{xy}$	$K' = 5b^2 D_x + 2a^2 D_{xy}$
$I = -30\frac{a^2}{b} D_y - 6bD_{xy}$	$N' = 15\frac{b^2}{a} D_x - 6aD_{xy}$
$K = 10b^2 D_x - 8a^2 D_{xy}$	$Z = 60\frac{b^2}{a^2} D_x + 30\frac{a^2}{b^2} D_y - 30D_1 - 84D_{xy}$
$L = 15\frac{b^2}{a} D_x - 15aD_1 - 6aD_{xy}$	$O' = -30\frac{b^2}{a^2} D_x - 30\frac{a^2}{b^2} D_y + 30D_1 + 84D_{xy}$
$O = 30\frac{b^2}{a} - 60\frac{a^2}{b^2} D_y - 30D_1 - 84D_{xy}$	

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When the shape functions are evaluated (Table 3.21) for a particular solid, line and plate element, the global coordinates and displacements at any point within the element are expressed in terms of the nodal values.

Table 3.22 gives the material compliance matrices for concrete, steel and timber. Table 3.23 gives details about loads and forces. The generalised nodal force equation is given below:

$$\{P\}^e = \left(\int_{\text{vol}} [B]^{T''}[D][B]\,\mathrm{d}V\right)\{u^e\} - \int_{\text{vol}} [B]^{T''}[D]\{\varepsilon_0\}\,\mathrm{d}V + \int_{\text{vol}} [B]^{T''}\{\sigma_0\}\,\mathrm{d}V - \int_S [N]^T\{p\}\,\mathrm{d}S - \int_{\text{vol}} [N]^{T''}\{G\}\,\mathrm{d}V \tag{3.194}$$

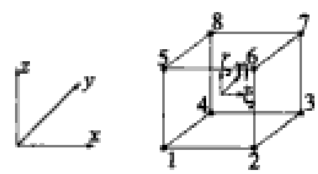
The terms given in Eq. (3.194) are defined in a matrix form.

The element stiffness matrix $[K]^e$ is then written as

$$[K]^e = \int_{\text{vol}} [B]^{T''}[D][B]\,\mathrm{d}V = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} B^{T''}DB\det\mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{3.195}$$

Table 3.21. Solid isoparametric elements (Bangash 1989, 1993).

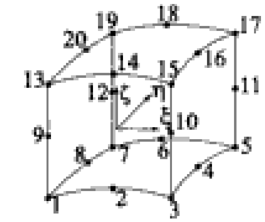
Eight-noded solid element



Node i	Shape functions $N_i(\xi,\eta,\zeta)$	Derivatives $\dfrac{\partial N_i}{\partial \xi}$	$\dfrac{\partial N_i}{\partial \eta}$	$\dfrac{\partial N_i}{\partial \zeta}$
1	$\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$	$-\frac{1}{8}(1-\eta)(1-\zeta)$	$-\frac{1}{8}(1-\xi)(1-\zeta)$	$-\frac{1}{8}(1-\eta)(1-\xi)$
2	$\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$	$\frac{1}{8}(1-\eta)(1-\zeta)$	$-\frac{1}{8}(1+\xi)(1-\zeta)$	$-\frac{1}{8}(1+\xi)(1-\eta)$
3	$\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$	$\frac{1}{8}(1+\eta)(1-\zeta)$	$\frac{1}{8}(1+\xi)(1-\zeta)$	$-\frac{1}{8}(1+\xi)(1+\eta)$
4	$\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$	$-\frac{1}{8}(1+\eta)(1-\zeta)$	$\frac{1}{8}(1-\xi)(1-\zeta)$	$-\frac{1}{8}(1-\xi)(1+\eta)$
5	$\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$	$-\frac{1}{8}(1-\eta)(1+\zeta)$	$-\frac{1}{8}(1-\xi)(1+\zeta)$	$\frac{1}{8}(1-\xi)(1-\eta)$
6	$\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$	$\frac{1}{8}(1-\eta)(1+\zeta)$	$-\frac{1}{8}(1+\xi)(1+\zeta)$	$\frac{1}{8}(1+\xi)(1-\eta)$
7	$\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$	$\frac{1}{8}(1+\eta)(1+\zeta)$	$\frac{1}{8}(1+\xi)(1+\zeta)$	$\frac{1}{8}(1+\xi)(1+\eta)$
8	$\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$	$-\frac{1}{8}(1+\eta)(1+\zeta)$	$\frac{1}{8}(1-\xi)(1+\zeta)$	$\frac{1}{8}(1-\xi)(1+\eta)$

Table 3.21 (cont.).

Twenty-noded solid element



Node i	Shape functions $N_i(\xi,\eta,\zeta)$	Derivatives $\dfrac{\partial N_i}{\partial \xi}$	$\dfrac{\partial N_i}{\partial \eta}$	$\dfrac{\partial N_i}{\partial \zeta}$
1	$\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)(-\xi-\eta-\zeta-2)$	$\frac{1}{8}(1-\eta)(1-\zeta)(2\xi+\eta+\zeta+1)$	$\frac{1}{8}(1-\xi)(1-\zeta)(2\eta+\xi+\zeta+1)$	$\frac{1}{8}(1-\xi)(1-\eta)(2\zeta+\eta+\xi+1)$
2	$\frac{1}{4}(1-\xi^2)(1-\eta)(1-\zeta)$	$-\frac{1}{2}(1-\eta)(1-\zeta)\xi$	$-\frac{1}{4}(1-\xi^2)(1-\zeta)$	$-\frac{1}{4}(1-\xi^2)(1-\eta)$
3	$\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)(\xi-\eta-\zeta-2)$	$\frac{1}{8}(1-\eta)(1-\zeta)(2\xi-\eta-\zeta-1)$	$\frac{1}{8}(1+\xi)(1-\zeta)(2\eta-\xi+\zeta+1)$	$\frac{1}{8}(1+\xi)(1-\eta)(2\zeta-\xi+\eta+1)$
4	$\frac{1}{4}(1+\xi)(1-\eta^2)(1-\zeta)$	$\frac{1}{4}(1-\eta^2)(1-\zeta)$	$-\frac{1}{2}(1+\xi)(1-\zeta)\eta$	$-\frac{1}{4}(1-\eta^2)(1+\xi)$
5	$\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)(\xi+\eta-\zeta-2)$	$\frac{1}{8}(1+\eta)(1-\zeta)(2\xi+\eta-\zeta-1)$	$\frac{1}{8}(1+\xi)(1-\zeta)(2\eta+\xi-\zeta-1)$	$\frac{1}{8}(1+\xi)(1+\eta)(2\zeta-\xi-\eta+1)$
6	$\frac{1}{4}(1-\xi^2)(1+\eta)(1-\zeta)$	$-\frac{1}{2}(1+\eta)(1-\zeta)\xi$	$\frac{1}{4}(1-\xi^2)(1-\zeta^2)$	$-\frac{1}{4}(1-\xi^2)(1+\eta)$
7	$\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)(-\xi+\eta-\zeta-2)$	$\frac{1}{8}(1+\eta)(1-\zeta)(2\xi-\eta+\zeta+1)$	$\frac{1}{8}(1-\xi)(1-\zeta)(2\eta-\xi-\zeta-1)$	$\frac{1}{8}(1-\xi)(1+\eta)(2\zeta-\eta+\xi+1)$
8	$\frac{1}{4}(1-\xi)(1-\eta^2)(1-\zeta)$	$-\frac{1}{2}(1-\eta^2)(1-\zeta)$	$-\frac{1}{2}(1-\xi)(1-\zeta)$	$-\frac{1}{4}(1-\eta^2)(1-\xi)$
9	$\frac{1}{4}(1-\xi)(1-\eta)(1-\zeta^2)$	$-\frac{1}{4}(1-\zeta^2)(1-\eta)$	$-\frac{1}{4}(1-\xi)(1-\zeta^2)$	$-\frac{1}{2}(1-\xi)(1-\eta)\zeta$
10	$\frac{1}{4}(1+\xi)(1-\eta)(1-\zeta^2)$	$\frac{1}{4}(1-\eta)(1-\zeta^2)$	$-\frac{1}{4}(1+\xi)(1-\zeta^2)$	$-\frac{1}{2}(1+\xi)(1-\eta)\zeta$
11	$\frac{1}{4}(1+\xi)(1+\eta)(1-\zeta^2)$	$\frac{1}{4}(1+\eta)(1-\zeta^2)$	$\frac{1}{4}(1+\xi)(1-\zeta^2)$	$-\frac{1}{2}(1+\xi)(1+\eta)\zeta$
12	$\frac{1}{4}(1-\xi)(1+\eta)(1-\zeta^2)$	$-\frac{1}{4}(1+\eta)(1-\zeta^2)$	$\frac{1}{4}(1-\xi)(1-\zeta^2)$	$-\frac{1}{2}(1-\xi)(1+\eta)\zeta$
13	$\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)(-\xi-\eta+\zeta-2)$	$\frac{1}{8}(1-\eta)(1+\zeta)(2\xi+\eta-\zeta+1)$	$\frac{1}{8}(1-\xi)(1+\zeta)(2\eta+\xi-\zeta+1)$	$\frac{1}{8}(1-\xi)(1-\eta)(2\zeta-\eta-\xi-1)$
14	$\frac{1}{4}(1-\xi^2)(1-\eta)(1+\zeta)$	$-\frac{1}{2}(1-\eta)(1+\zeta)\xi$	$-\frac{1}{4}(1-\xi^2)(1+\zeta)$	$\frac{1}{4}(1-\xi^2)(1-\eta)$
15	$\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)(\xi-\eta+\zeta-2)$	$\frac{1}{8}(1-\eta)(1+\zeta)(2\xi-\eta+\zeta-1)$	$\frac{1}{8}(1+\xi)(1+\zeta)(2\eta-\xi-\zeta+1)$	$\frac{1}{8}(1-\eta)(1+\xi)(2\zeta+\xi-\eta-1)$
16	$\frac{1}{4}(1+\xi)(1-\eta^2)(1+\zeta)$	$\frac{1}{4}(1-\eta^2)(1+\zeta)$	$-\frac{1}{2}(1+\xi)(1+\zeta)\eta$	$\frac{1}{4}(1+\xi)(1-\eta^2)$
17	$\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)(\xi+\eta+\zeta-2)$	$\frac{1}{8}(1+\eta)(1+\zeta)(2\xi+\eta+\zeta-1)$	$\frac{1}{8}(1+\xi)(1+\zeta)(2\eta+\xi+\zeta-1)$	$\frac{1}{8}(1+\xi)(1+\eta)(2\zeta+\eta+\xi-1)$
18	$\frac{1}{4}(1-\xi^2)(1+\eta)(1+\zeta)$	$-\frac{1}{2}(1+\eta)(1+\zeta)$	$\frac{1}{4}(1-\xi^2)(1+\zeta)$	$\frac{1}{4}(1-\xi^2)(1+\eta)$
19	$\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)(-\xi+\eta+\zeta-2)$	$\frac{1}{8}(1+\eta)(1+\zeta)(2\xi-\eta-\zeta-1)$	$\frac{1}{8}(1-\xi)(1+\zeta)(2\eta-\xi+\zeta-1)$	$\frac{1}{8}(1-\xi)(1+\eta)(2\zeta-\xi+\eta-1)$
20	$\frac{1}{4}(1-\xi)(1-\eta^2)(1+\zeta)$	$-\frac{1}{4}(1-\eta^2)(1+\zeta)$	$-\frac{1}{2}(1-\xi)(1+\zeta)\eta$	$\frac{1}{4}(1-\xi)(1-\eta^2)$

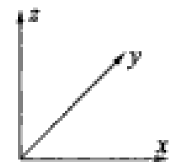
Table 3.21 (cont.).

Two, three and four-noded elements

The shape functions and derivatives for the isoparametric line elements are given below.

a) Two-noded line element

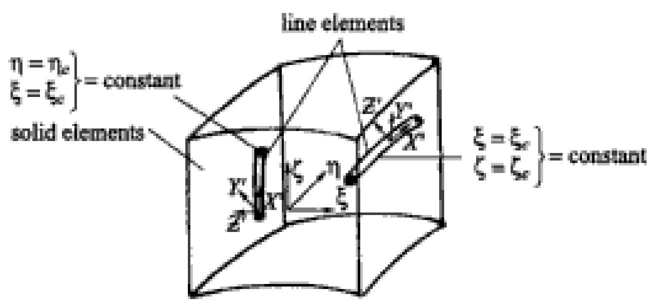
Shape functions	Derivatives
$N_1 = \frac{1}{2}(1-\xi)$	$\frac{\partial N_1}{\partial \xi} = -\frac{1}{2}$
$N_2 = \frac{1}{2}(1+\xi)$	$\frac{\partial N_2}{\partial \xi} = \frac{1}{2}$



Global axes

b) Three-noded line element

Shape functions	Derivatives
$N_1 = \frac{1}{2}(\xi-1)\xi$	$\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2}$
$N_2 = 1-\xi^2$	$\frac{\partial N_2}{\partial \xi} = 2\xi$
$N_3 = \frac{1}{2}(\xi+1)\xi$	$\frac{\partial N_3}{\partial \xi} = \xi + \frac{1}{2}$



c) Four-noded line element

Shape functions	Derivatives
$N_1 = \frac{1}{3}(1-\xi)\left(2\xi^2 - \frac{1}{2}\right)$	$\frac{\partial N_1}{\partial \xi} = \frac{1}{3}\left(4\xi - 6\xi^2 + \frac{1}{2}\right)$
$N_2 = \frac{4}{3}(\xi^2-1)\left(\xi - \frac{1}{2}\right)$	$\frac{\partial N_2}{\partial \xi} = \frac{4}{3}\left(3\xi^2 - \xi - 1\right)$
$N_3 = \frac{4}{3}(1-\xi^2)\left(\xi + \frac{1}{2}\right)$	$\frac{\partial N_3}{\partial \xi} = \frac{4}{3}\left(1 - 3\xi^2 - \xi\right)$
$N_4 = \frac{1}{3}(1+\xi)\left(2\xi^2 - \frac{1}{2}\right)$	$\frac{\partial N_4}{\partial \xi} = \frac{1}{3}\left(4\xi + 6\xi^2 - \frac{1}{2}\right)$

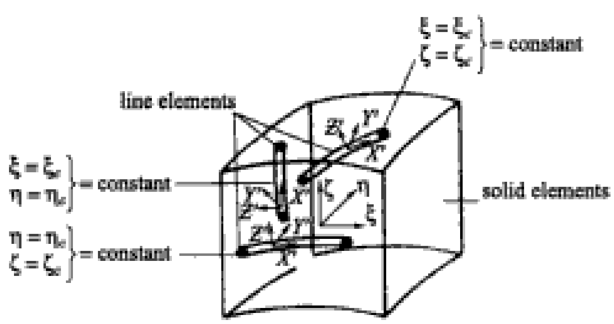


Table 3.22. Material compliance matrices (Bangash 1989, 1993).

a) Concrete or composite material

$[D]$ – Variable Young's modulus and Poisson's ratio

$$\begin{bmatrix} D_{11} = \frac{(1-v_{23}v_{32})}{\bar{v}}E_1 & D_{12} = \frac{(v_{12}+v_{12}v_{32})}{\bar{v}}E_2 & D_{13} = \frac{(v_{13}+v_{12}v_{23})}{\bar{v}}E_3 & D_{14}=0 & D_{15}=0 & D_{16}=0 \\ D_{21} = \frac{(v_{21}+v_{23}v_{31})}{\bar{v}}E_1 & D_{22} = \frac{(1-v_{13}v_{31})}{\bar{v}}E_2 & D_{23} = \frac{(v_{23}+v_{13}v_{21})}{\bar{v}}E_3 & D_{24}=0 & D_{25}=0 & D_{26}=0 \\ D_{31} = \frac{(v_{31}+v_{21}v_{32})}{\bar{v}}E_1 & D_{32} = \frac{(v_{32}+v_{12}v_{31})}{\bar{v}}E_2 & D_{33} = \frac{(1-v_{12}v_{21})}{\bar{v}}E_3 & D_{34}=0 & D_{35}=0 & D_{36}=0 \\ D_{41}=0 & D_{42}=0 & D_{43}=0 & D_{44}=0 & D_{45}=0 & D_{46}=0 \\ D_{51}=0 & D_{52}=0 & D_{53}=0 & D_{54}=0 & D_{55}=0 & D_{56}=0 \\ D_{61}=0 & D_{62}=0 & D_{63}=0 & D_{64}=0 & D_{65}=0 & D_{66}=0 \end{bmatrix}$$

$$\bar{v} = 1 - v_{12}v_{21} - v_{13}v_{31} - v_{23}v_{32} - v_{12}v_{23}v_{31} - v_{21}v_{13}v_{32}$$

$$E_1 v_{21} = E_2 v_{12} \quad D_{55} = G_{23}$$

$$E_2 v_{32} = E_3 v_{23} \quad D_{66} = G_{13}$$

$$E_3 v_{13} = E_1 v_{31}$$

The values of G_{12}, G_{23} and G_{13} are calculated in terms of modulus of elasticity and Poisson's ratio as follows:

$$G_{12} = \frac{1}{2}\left[\frac{E_1}{2(1+v_{12})} + \frac{E_2}{2(1+v_{21})}\right] = \frac{1}{2}\left[\frac{E_1}{2(1+v_{12})} + \frac{E_1}{2\left(\frac{E_1}{E_2}+v_{12}\right)}\right]$$

$$G_{23} = \frac{1}{2}\left[\frac{E_2}{2(1+v_{23})} + \frac{E_3}{2(1+v_{32})}\right] = \frac{1}{2}\left[\frac{E_2}{2(1+v_{23})} + \frac{E_2}{2\left(\frac{E_2}{E_3}+v_{23}\right)}\right]$$

$$G_{13} = \frac{1}{2}\left[\frac{E_3}{2(1+v_{31})} + \frac{E_1}{2(1+v_{13})}\right] = \frac{1}{2}\left[\frac{E_3}{2(1+v_{31})} + \frac{E_3}{2\left(\frac{E_3}{E_1}+v_{31}\right)}\right]$$

b) For isotropic cases: $E_1 = E_2 = E_3 = E$

c) Steel or timber: $v_{12} = v_{13} = v_{23} = v_{21} = v_{31} = v_{32} = v$
$[D]$ – Constant Young's modulus and Poisson's ratio

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

Table 3.23. Miscellaneous loads and force (Bangash 1989).

Gravitational forces (surface forces)

Equivalent nodal force in the line of gravity Z direction.

$$\{P_z\}_i = \int_v [N^T]_i \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} \mathrm{d\,Vol} = \sum_{i=1}^{n}\sum_{i=1}^{n}\sum_{i=1}^{n} [N]^T_{i,j,K} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} |J|_{i,j,K} W_1 W_j W_K$$

Body forces

Body force component per unit volume at $(\bar{X}, \bar{Y})$ point is:

$$\begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \{\bar{f}\} = P_b w^2 \begin{Bmatrix} X \\ Y \\ 0 \end{Bmatrix}, \qquad 0 \neq \int_v [N]^T \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \mathrm{d\,vol}$$

In the case of isoparametric elements.

Concentrated loads

Concentrated loads away from the point.

$$\{P_{e0}\} = N_i(\xi_1, \eta_1, \zeta_1)P, \quad \xi_1 = \xi, \quad \eta_1 = -\eta, \quad \zeta = +1$$

Distributed loads

$$\{P_s\} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} N_i^{T''}[p_x, p_y, p_z]^{T''} \begin{Bmatrix} \frac{\partial Y}{\partial \xi} & \frac{\partial Z}{\partial \eta} & -\frac{\partial Z}{\partial \xi} & \frac{\partial Y}{\partial \eta} \\ \frac{\partial Z}{\partial \xi} & \frac{\partial X}{\partial \eta} & -\frac{\partial Z}{\partial \eta} & \frac{\partial X}{\partial \xi} \\ \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \eta} & -\frac{\partial Y}{\partial \xi} & \frac{\partial X}{\partial \eta} \end{Bmatrix} \mathrm{d}\xi\, \mathrm{d}\eta, \quad \text{for } \zeta = \pm 1$$

similarly for $\xi = \pm 1$, $\eta = \pm 1$.

Thermal loads

$$\{P\}_T = \int B^T D\varepsilon_T \,\mathrm{d}v = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} [B]_{i,j,K}[D]\{\varepsilon_T\}|J|_{i,j,K} W_i W_j W_K$$

$$\{\varepsilon_T\} = \left[\alpha_T T, \alpha_T T, \alpha_T T \;\; 0\;0\;0\right]^T, \quad T = \sum_{i=1}^{n} N_i T_i$$

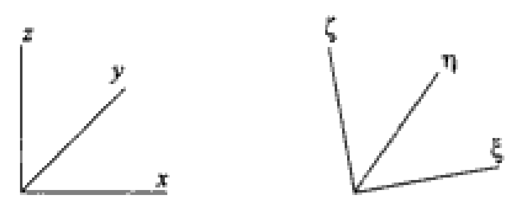
Creep loads

$$\{P\}_{cv} = \int B^T D\varepsilon_c \,\mathrm{d}v = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} [B]_{i,j,K}[D]\{\varepsilon_c\}_{i,j,K} W_i W_j W_K \,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

Table 3.24. Stress and strain transformation matrices (Bangash 1989).

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

$\{T''_\sigma\}, \{T''_\varepsilon\}$



Global axis Local axis

Table 3.24 (cont.).

Direction cosines of the two axes are given by:

$$l_1 = \cos(\xi, x), \quad m_1 = \cos(\xi, y), \quad n_1 = \cos(\xi, z)$$
$$l_2 = \cos(\eta, x), \quad m_2 = \cos(\eta, y), \quad n_2 = \cos(\eta, z)$$
$$l_3 = \cos(\zeta, x), \quad m_3 = \cos(\zeta, y), \quad n_3 = \cos(\zeta, z)$$

The following relationships can be written for local and global strain and stress vectors:

$$\{\varepsilon'_x\} = [T_\varepsilon]\{\varepsilon_x\}, \qquad \{\sigma'_x\} = [T_\varepsilon]^T\{\sigma'_x\}$$

and also

$$\{\sigma'_x\} = [T_\sigma]\{\sigma_x\}, \qquad \{\varepsilon_x\} = [T_\sigma]^T\{\varepsilon'_x\}$$

$$[T''_\varepsilon] = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1m_1 & m_1n_1 & l_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2m_2 & m_2n_2 & l_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3m_3 & m_3n_3 & l_3n_3 \\ 2l_1l_2 & 2m_1m_2 & 2n_1n_2 & l_1m_2+l_2m_1 & m_1n_2+m_2n_1 & l_1n_2+l_2n_1 \\ 2l_2l_3 & 2m_2m_3 & 2n_2n_3 & l_2m_3+l_3m_2 & m_2n_3+m_3n_2 & l_2n_3+l_3n_2 \\ 2l_1l_3 & 2m_1m_3 & 2n_1n_3 & l_1m_3+m_1l_3 & m_1n_3+m_3n_1 & l_1n_3+n_1l_3 \end{bmatrix}$$

$$[T''_\sigma] = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & 2l_1m_1 & 2m_1n_1 & 2l_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2m_2n_2 & 2l_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2m_3n_3 & 2l_3n_3 \\ l_1l_2 & m_1m_2 & n_1n_2 & l_1m_2+l_2m_1 & m_1n_2+m_2n_1 & l_1n_2+l_2n_1 \\ l_2l_3 & m_2m_3 & n_2n_3 & l_2m_3+l_3m_2 & m_2n_3+m_3n_2 & l_2n_3+l_3n_2 \\ l_1l_3 & m_1m_3 & n_1n_3 & l_1m_3+l_3m_1 & m_1n_3+m_3n_1 & l_1n_3+l_3n_1 \end{bmatrix}$$

The nodal force due to body force:

$$\{P_b\}^e = \int_{\text{vol}} [N]^{T''}\{G\}\,\mathrm{d}V = -\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} N^{T''} G \det \mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \qquad (3.196)$$

The nodal force due to surface force:

$$\{P_s\}^e = -\int_S [N]^{T''}\{p\}\,\mathrm{d}S \qquad (3.197)$$

The nodal force due to initial stress:

$$\{P_{\sigma_0}\}^e = \int_{\text{vol}} [B]^{T''}\{\sigma_0\}\,\mathrm{d}V = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} B^{T''}\sigma_0 \det \mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \qquad (3.198)$$

The nodal force due to initial strain:

$$\{P_{\varepsilon_0}\}^e = \int_{\text{vol}} [B]^{T''}[D]\{\varepsilon_0\}\,\mathrm{d}V$$
$$= -\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} B^{T''} D\varepsilon_0 \det \mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{3.199}$$

Equation (3.195) can be written as:

$$\{F\}^e = [K]^e\{u\}^e + \{P_b\}^e + \{P_s\}^e + \{P_{\sigma_0}\}^e + \{P_{\varepsilon_0}\}^e \tag{3.200}$$

Where shear and torsion are to be involved, Table 3.23 gives the stiffness matrix $[K]$ which is included in the overall stiffness matrix $[K]$ of the structure and in this case the stairs. During the finite element analysis, sometimes it becomes necessary to transform stress and strain matrices $\{T_\delta''\}$ and $\{T_\varepsilon''\}$, respectively, from local axes to global axes or vice versa. These are done with the help of Table 3.24. The finite element mesh schemes are given in Figures 3.50 and 3.51.

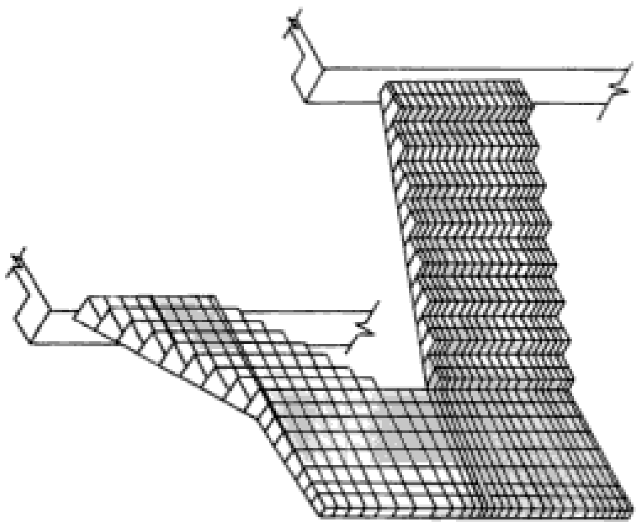


Figure 3.50. Mesh scheme for an integrated staircase using isoparametric elements.

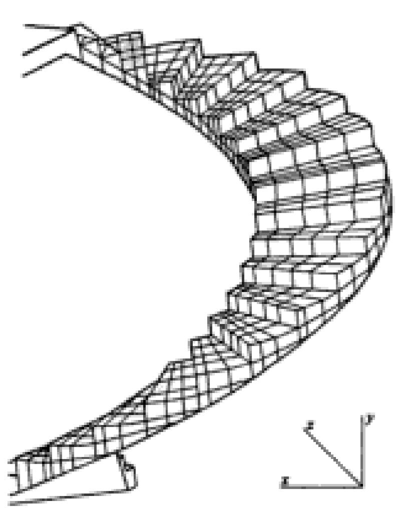


Figure 3.51. Three-dimensional isoparametric element mesh scheme for a helical staircase.

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Table 3.25 (cont.).

the values of C_1, C_2 and C_3 are defined as

$$C_1 = \frac{\partial J_1}{\partial\{\sigma\}} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = S_x \begin{Bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + S_y \begin{Bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + S_z \begin{Bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + 2 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{h}$$

$$C_2 = \frac{\partial J_2}{\partial\{\sigma\}} = \frac{\partial J_2}{\partial S_x}\frac{\partial S_x}{\partial\{\sigma\}} + \frac{\partial J_2}{\partial S_y}\frac{\partial S_y}{\partial\{\sigma\}} + \frac{\partial J_2}{\partial S_z}\frac{\partial S_z}{\partial\{\sigma\}}$$
$$+ \frac{\partial J_2}{\partial \tau_{xy}}\frac{\partial \tau_{xy}}{\partial\{\sigma\}} + \frac{\partial J_2}{\partial \tau_{yz}}\frac{\partial \tau_{yz}}{\partial\{\sigma\}} + \frac{\partial J_2}{\partial \tau_{zx}}\frac{\partial \tau_{zx}}{\partial\{\sigma\}}$$

In a matrix form C_2 is given as:

$$C_2 = \begin{Bmatrix} \frac{1}{3}(2S_x - S_y - S_z) \\ \frac{1}{3}(2S_y - S_x - S_z) \\ \frac{1}{3}(2S_z - S_x - S_y) \\ 2\tau_{xy} \\ 2\tau_{yz} \\ 2\tau_{zx} \end{Bmatrix} = \begin{Bmatrix} S_x \\ S_y \\ S_z \\ 2\tau_{xy} \\ 2\tau_{yz} \\ 2\tau_{zx} \end{Bmatrix} \tag{i}$$

$$\cos 3\theta = \frac{3\sqrt{3}}{2} \times \frac{J_3}{J_2^{3/2}} \tag{j}$$

$$C_3 = \frac{\partial \cos 3\theta}{\partial J_3} = \frac{\partial \cos 3\theta}{\partial J_3}\frac{\partial J_3}{\partial\{\sigma\}} + \frac{\partial \cos 3\theta}{\partial J_2}\frac{\partial J_2}{\partial\{\sigma\}} \tag{k}$$

from Eq. (k)

$$\frac{\partial \cos 3\theta}{\partial J_3} = \frac{3\sqrt{3}}{2J_2^{3/2}}, \qquad \frac{\partial \cos 3\theta}{\partial J_2} = \left(\frac{3\sqrt{3}}{2}J_3\right)\left(\frac{-\frac{3}{2}}{J_2^{5/2}}\right) = -\frac{9}{4}\sqrt{3}\frac{J_3}{J_2^{5/2}} \tag{l}$$

hence

$$C_3 = \frac{3}{2}\frac{\sqrt{3}}{J_2^{3/2}}\frac{\partial J_3}{\partial\{\sigma\}} - \frac{9}{4}\sqrt{3}\frac{J_3}{J_2^{5/2}}\frac{\partial J_2}{\partial\{\sigma\}} = \frac{3}{2}\frac{\sqrt{3}}{J_2^{3/2}}\left[\frac{\partial J_3}{\partial\{\sigma\}} - \frac{3}{2}\frac{J_3}{J_2}\frac{\partial J_2}{\partial\{\sigma\}}\right] \tag{m}$$

now

$$\frac{\partial J_3}{\partial S_x} = S_y S_z - \tau_{yz}^2, \qquad \frac{\partial J_3}{\partial S_y} = S_x S_z - \tau_{xz}^2, \qquad \frac{\partial J_3}{\partial S_z} = S_x S_y - \tau_{xy}^2 \tag{n}$$

$$S_x = \frac{1}{3}(2\sigma_x - \sigma_y - \sigma_z) \tag{o}$$

$$\frac{\partial S_x}{\partial\{\sigma\}} = \frac{1}{3}\begin{Bmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \frac{\partial S_y}{\partial\{\sigma\}} = \frac{1}{3}\begin{Bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \frac{\partial S_z}{\partial\{\sigma\}} = \frac{1}{3}\begin{Bmatrix} -1 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{p}$$

Table 3.25 (cont.).

$$\frac{\partial J_3}{\partial \tau_{xy}} = 2\tau_{yz}\tau_{zx} - 2S_z\tau_{xy}, \qquad \frac{\partial J_3}{\partial \tau_{yz}} = 2\tau_{xy}\tau_{zx} - 2S_x\tau_{yz}, \qquad \frac{\partial J_3}{\partial \tau_{xz}} = 2\tau_{xy}\tau_{yz} - 2S_y\tau_{xz} \tag{q}$$

$$\frac{\partial \tau_{xy}}{\partial\{\sigma\}} = [000100]^T, \qquad \frac{\partial \tau_{yz}}{\partial\{\sigma\}} = [000010]^T, \qquad \frac{\partial \tau_{zx}}{\partial\{\sigma\}} = [000001]^T \tag{r}$$

$$\frac{\partial J_3}{\partial\{\sigma\}} = \begin{Bmatrix} \frac{1}{3}\left[2\left(S_yS_z - \tau_{yz}^2\right) - \left(S_xS_z - \tau_{xz}^2\right) - \left(S_xS_y - \tau_{xy}^2\right)\right] \\ \frac{1}{3}\left[-\left(S_yS_z - \tau_{yz}^2\right) + 2\left(S_xS_z - \tau_{xz}^2\right) - \left(S_xS_y - \tau_{xy}^2\right)\right] \\ \frac{1}{3}\left[-\left(S_yS_z - \tau_{yz}^2\right) - 2\left(S_xS_z - \tau_{xz}^2\right) + 2\left(S_xS_y - \tau_{xy}^2\right)\right] \\ 2(\tau_{yz}\tau_{zx} - S_z\tau_{xy}) \\ 2(\tau_{xy}\tau_{zx} - S_x\tau_{yz}) \\ 2(\tau_{xy}\tau_{yz} - S_y\tau_{xz}) \end{Bmatrix} \tag{s}$$

Equation (s) is further simplified as:

$$\frac{\partial J_3}{\partial\{\sigma\}} = \begin{Bmatrix} \frac{1}{3}\left[2S_yS_z - S_xS_z - S_xS_y - 2\tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2\right] \\ \frac{1}{3}\left[2S_xS_z - S_yS_z - S_xS_y - 2\tau_{xz}^2 + \tau_{yz}^2 + \tau_{xy}^2\right] \\ \frac{1}{3}\left[2S_xS_y - S_yS_z - S_xS_z - 2\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right] \\ 2(\tau_{yz}\tau_{zx} - S_z\tau_{xy}) \\ 2(\tau_{xy}\tau_{zx} - S_x\tau_{yz}) \\ 2(\tau_{xy}\tau_{yz} - S_y\tau_{xz}) \end{Bmatrix} \tag{sa}$$

From the flow rule of the normality principle, the following relationship exists between the plastic strain increment and the plastic stress increment:

$$\mathrm{d}\{\varepsilon\} = \lambda\frac{\partial f}{\partial\{\sigma\}} \tag{t}$$

This equation can be interpreted as requiring the normality of the plastic strain increment vector to yield the surface in the hyper space of n stress dimensions. As before, $\mathrm{d}\lambda$ is proportionality constant.

For stress increments of infinitesimal size, the change of strain can be divided into elastic and plastic parts, thus (as before):

$$\mathrm{d}\{\varepsilon\} = \mathrm{d}\{\varepsilon\}_e + \mathrm{d}\{\varepsilon\}_p \tag{u}$$

The elastic increment of stress and strain is related to an isotropic material property matrix

$$[D] \quad \text{by} \quad \mathrm{d}\{\varepsilon\}_e[D]^{-1}\{\mathrm{d}\sigma\} \tag{v}$$

$$\mathrm{d}\{\varepsilon\}_e = [D]^{-1}\,\mathrm{d}\{\sigma\} + \mathrm{d}\lambda\left\{\frac{\partial f}{\partial \sigma}\right\} \tag{w}$$

The function stresses, on differentiation, can be written as:

$$\mathrm{d}f = \frac{\partial f}{\partial \sigma_1}\mathrm{d}\sigma_1 + \frac{\partial f}{\partial \sigma_2}\mathrm{d}\sigma_2 + \cdots \qquad 0 = \left\{\frac{\partial f}{\partial \sigma}\right\}^T \mathrm{d}\{\sigma\} \tag{x}$$

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Table 3.25 (cont.).

```
 3800 SUB1=PK-PAR(JP,1)
      SUB2=PAR(JP+1,1)-PAR(JP,1)
      A=SUB1*(PAR(JP+1,2)-PAR(JP,2))/SUB2+PAR(JP,2)
      B=SUB1*(PAR(JP+1,3)-PAR(JP,3))/SUB2+PAR(JP,3)
      PK1=SUB1*(PAR(JP+1,4)-PAR(JP,4))/SUB2+PAR(JP,4)
      PK2=SUB1*(PAR(JP+1,5)-PAR(JP,5))/SUB2+PAR(JP,5)
 3900 VARI1=SIG(1)+SIG(2)+SIG(3)
      VARJ2=1.0/6.0*((SIG(1)-SIG(2))**2+(SIG(2)-SIG(3))**2+
     @     (SIG(3)-SIG(1))**2+SIG(4)**2+SIG(5)**2+SIG(6)**2
      VARI13=VARI1/3.0
      VII31=SIG(1)-VARI13
      VII32=SIG(2)-VARI13
      VII33=SIG(3)-VARI13
      VARJ3=VII31*(VII32*VII33-SIG(5)**2)-SIG(4)*SIG(4)*VII33
     @       -SIG(5)*SIG(5))+SIG(6)*(SIG(4)*SIG(5)-SIG(6)*VII32)
      VAR3TH=1.5*3.0**(0.5)*VARJ3/VARJ2**1.5
      IF(VAR3TH .GE. 0.0) GOTO 4000
      ALAM=22.0/21.0-1.0/3.0*ACOS(-PK2*VAR3TH)
      TOTLAM=PK1*COS(ALAM)
      DFD3TH=PK1*PK2*VARJ2**0.5*SIN(ALAM)/(3.0*PROP(4)*
     @        SIN(ACOS(-PK2*VAR3TH)))
      GOTO 4100
 4000 ALAM=1.0/(3.0*ACOS(PK2*VAR3TH)
      TOTLAM=PK1*COS(ALAM)
      DFD3TH=PK1*PK2*VARJ2**0.5*SIN(ALAM)/(3.0*PROP(4)*
     @       SIN(ACOS(PK2*VAR3TH)))
 4100 DFDI1=B/PROP(4)
      DFDJ2=A/PROP(4)**2+TOTLAM/(PROP(4)*VARJ2**0.5)
      DVI1DS(1)=1.0
      DVI1DS(2)=1.0
      DVI1DS(3)=1.0
      DVI1DS(4)=0.0
      DVI1DS(5)=0.0
      DVI1DS(6)=0.0
      DVJ2DS(1)=1.0/3.0*(2.0*SIG(1)-SIG(2)-SIG(3))
      DVJ2DS(2)=1.0/3.0*(2.0*SIG(2)-SIG(1)-SIG(3))
      DVJ2DS(3)=1.0/3.0*(2.0*SIG(3)-SIG(1)-SIG(2))
      DVJ2DS(4)=2.0*SIG(4)
      DVJ2DS(5)=2.0*SIG(5)
      DVJ2DS(6)=2.0*SIG(6)
      DVJ3DS(1)=1.0/3.0*(VII31*(-VII32-VII33))+2.0*VII32*VII31-
     @           2.0*SIG(5)**2+SIG(4)**2+SIG(6)**2
      DVJ3DS(2)=1.0/3.0*(VII32*(-VII31-VII33))+2.0*VII31*VII33-
     @           2.0*SIG(6)**2+SIG(4)**2+SIG(5)**2
      DVJ3DS(3)=1.0/3.0*(VII33*(-VII31-VII32))+2.0*VII31*VII32-
     @           2.0*SIG(4)**2+SIG(5)**2+SIG(6)**2
      DVJ3DS(4)=-2.0*VII33*SIG(4)+2.0*SIG(5)*SIG(6)
      DVJ3DS(5)=-2.0*VII31*SIG(5)+2.0*SIG(4)*SIG(6)
```

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4. Accumulate total displacements

$$\{U_i\} = \{U_{i-1}\} + \{\Delta U_i\} \tag{3.212}$$

5. Calculate strain increments as,

$$\{\Delta\varepsilon_i\} = [B]\{\Delta U_i\} \tag{3.213}$$

and strains

$$\{\varepsilon_i\} = \{\varepsilon_{i-1}\} + \{\Delta\varepsilon_i\} \tag{3.214}$$

6. The stress increments are calculated using the current non-linear constitutive matrices

$$\{\Delta\sigma_i\} = \{f(\sigma)\}\{\Delta\varepsilon\} \tag{3.215}$$

Accumulate stresses as:

$$\{\sigma_i\} = \{\sigma_{i-1}\} + \{\Delta\sigma_i\} \tag{3.216}$$

ISP-stress point indictor

$= 0$ elastic point
$= 1$ plastic point
$= 2$ unloaded from plastic state
$= \sigma_y$ uniaxial yield stress

6.1 Firstly, the stress increment is calculated using the elastic material matrix as $\{\Delta\sigma_i'\} = [D]_e^s\{\Delta\varepsilon\}$ where $[D]_e^s$ is the elastic material matrix for any material in the staircase. First estimate of total stress:

$$\{\sigma_i'\} = \{\sigma_{i-1}\} - \{\Delta\sigma_i'\} \tag{3.217}$$

6.2 Calculate

$$\{\bar{\sigma}_i\} = \left\{f(\sigma_i')\right\}_1\{\bar{\sigma}_{i-1}\} = \left\{f(\sigma_{i-1})\right\} \text{ (von Mises)} \tag{3.218}$$

yield criterion or other yield criterion such as Ottoson etc.
6.3 If plastic point (i.e. ISP $= 1$), go to step 6.5.
6.4 $\bar{\sigma}_i \geqslant \sigma_y$ point (ISP $= 1$), transition from elastic to plastic, calculate factor f_{TR}

$$f_{TR} = \left(\frac{\sigma_y - \bar{\sigma}_{i-1}}{\bar{\sigma}_i - \bar{\sigma}_{i-1}}\right) \quad \text{(see Figures 3.52 and 3.53)} \tag{3.219}$$

stress at yield surface

$$\{\sigma_i\}Y = (\sigma_{i-1}) + f_{TR}^*\{\Delta\sigma_i'\} \tag{3.220}$$

calculate elasto-plastic stress increment

$$\{\Delta\sigma_i\} = [D]_{ep}^s\{\sigma_i\}^{Y^*}(1 - f_{TR})\{\Delta\varepsilon\} \tag{3.221}$$

total stress

$$\{\sigma_i\} + \{\sigma_i\}^Y + \{\Delta\sigma_i\} \tag{3.222}$$

set ISP = 1; go to 6.7

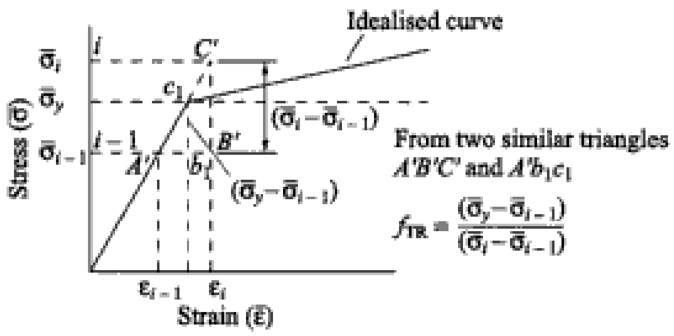
if $\bar{\sigma}_i < \sigma_y$ elastic point, $\sigma_i = \sigma'_i$ go to step 6.8.

6.5 Plastic point in the previous iteration, check for unloading i.e. $\bar{\sigma}_i \geqslant \sigma_y$, see Figure 3.53 go to step 6.6.

Unloading at this point, set ISP = 2, total stress

$$\{\sigma_i\} = \{\sigma_{i-1}\} + \{\Delta\sigma_i\} \text{ and set } \{\sigma_y\} = \{\bar{\sigma}_{i-1}\} \tag{3.323}$$

go to step 6.8.



a) Equivalent stress-strain curve for steel

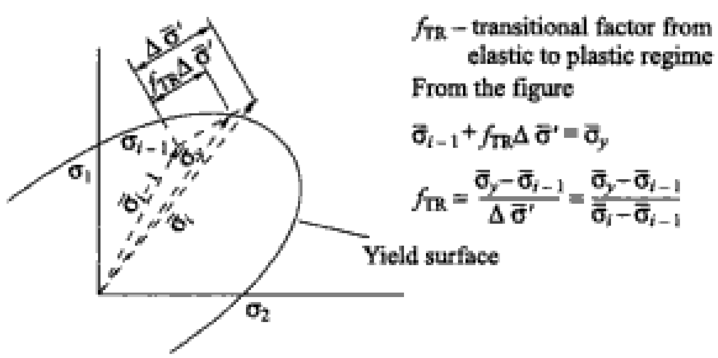


Figure 3.52. Yield criteria.

b) Yield surface in the principal stress axes

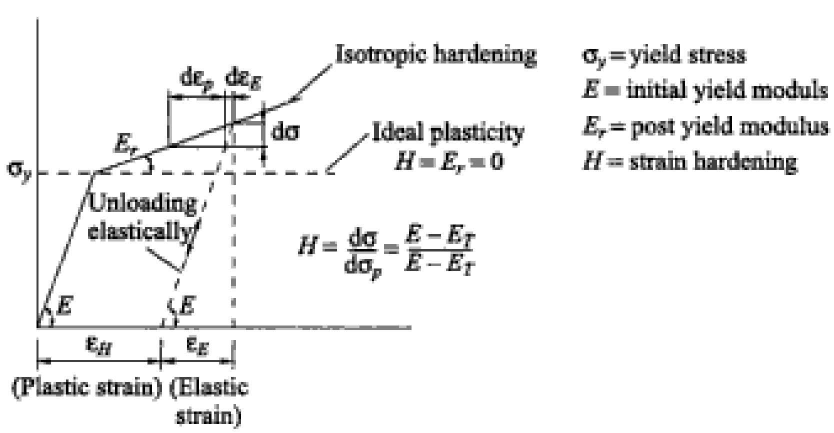


Figure 3.53. Stress-strain curve for steel plate with elastic unloading.

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CHAPTER 4

Staircases and their analyses: A comparative study

4.1 INTRODUCTION

Various analyses mentioned in the text are examined in this chapter. A total number of one hundred and fifty free-standing stairs and thirty helical or horseshoe or a combination of these are examined. Two types of finite element analysis are carried out. One particular analysis is based on pure bending, shear and displacement using polynomials of a specific order. Where torsion and axial effects are to be included, isoparametric finite element analysis is adopted in which a provision is made for solid elements representing concrete and steel major sections and line elements representing reinforcement; either matched with the nodes of solid element or placed in the body of the solid elements. For steel stairs, the same finite element analysis is adopted except where plates are used; a special plate element is given together with a displacement polynomial in Appendix 1. For isoparametric finite element analysis various shape functions are included in a specially developed computer program, ISOPAR. The output gives stresses, strains, plasticity index, cracks, failure modes, steel yielding and fracture under static, dynamic and impact loads.

4.2 A COMPARATIVE STUDY OF RESULTS

4.2.1 *Free-standing stairs*

It is interesting to review the assumptions made in some of the analyses. Liebenberg (1956, 1960, 1962) has introduced the concept of the space interaction of plates in order to analyse free-standing staircases. This means that the staircase can be treated as an indeterminate structure. No torsional effects are included. Siev (1962, 1963) extended Liebenberg's method to include secondary stress resulting from the compatibility conditions at the intersection between the landing and the flights. Torsional moments and their stresses, being negligible, are taken as secondary stresses in order to evaluate primary stresses. Gould (1963) and Taleb (1964) have produced simplified analyses by neglecting the bending moments along the line of intersection of the landing and the flights. Cusens and Kuang (1966) have assumed that the staircase behaviour can be simulated by the skeletal rigid frame. Two halves of the staircase are then taken as determinate structures and horizontal restraining forces and moments are applied on each half. As a result bending and torsional moments, axial and shearing forces are evaluated. This concept is well recognised and is extremely valuable. As can be seen, the results are obtained by the program ISOPAR. The number of solid elements and line elements is 250 and 1200, respectively. Based on the model adopted by Cusens and Kuang (1966) and as shown in Figure 4.1, finite element analysis using the isoparametric elements representing concrete elements and line elements in the body of these solid elements give a failure load factor (excessive cracking, bursting of the reinforcement and the dislocation of the landing from the flights) of 7.1 against the experimental value of 6.48.

The same mesh, for economic reasons, is kept in the finite element analysis when the results of others have been investigated. The loadings, dimensions and others including boundary conditions assumed by the authors are included in the finite element analysis; except that the torsional aspect is not ignored. Figure 4.2 shows the comparative study of results of the finite element analysis with those used from the analysis produced by various authors (Kersten and Kuhnert 1957, Atrops 1966, Cusen and Kuang 1966, Leonhardt and Monnig 1973, 1975 and Bangash 1993).

A finite element analysis was carried out for slabless tread-riser stairs. This time the analysis was based on the elasticity of the materials used in such stairs. The far ends of such stairs are assumed to be fixed. No failure analysis was carried out and the stairs were analysed within the design limits. The analysis is in line with other research so that it can be compared easily with the simplified analysis produced by Figure 4.3, showing the comparison between the two analyses for stairs with different number of steps for the ratio of width of tread/height of riser $\overline{G}/H_1$ ranging from 0.4 to 1.0.

A plate-bending finite element analysis was adopted for steps. Next 'Scissors' type staircases were analysed. Since isoparametric finite element analyses are not involved directly in producing bending moments, it was necessary initially to adopt the plate-bending finite element analysis. Bending stresses and shear stresses were produced for various positions of loadings. Using the same mesh size, a two dimensional isoparametric finite element was carried out assuming the same boundary conditions. Stresses and strain were produced. In most places they were almost the same and the errors in them were within 5.5%. The isoparametric parametric finite element analysis was then extended to include torsional and axial effects. Figures 4.4 and 4.5 give parameters for moments and axial forces for various widths of such stairs. The following gives further explanations for their use. A typical example is given which is based

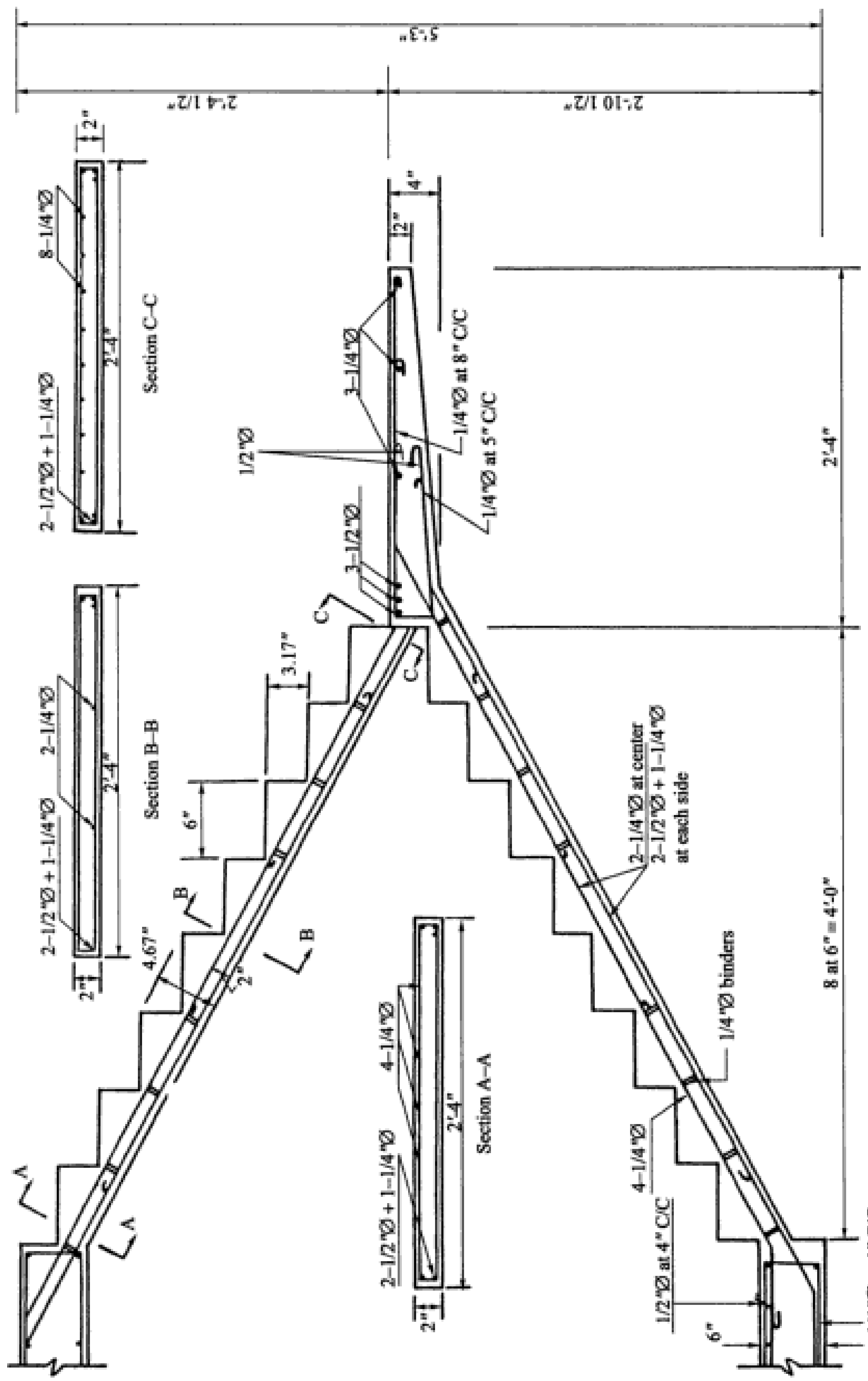
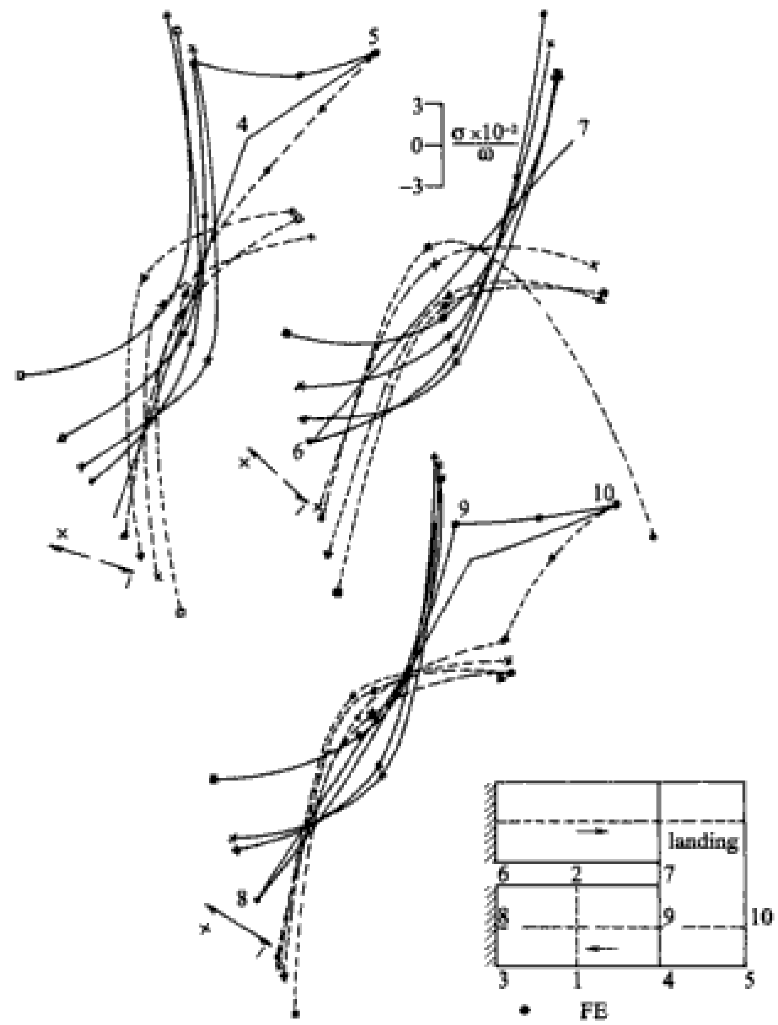


Figure 4.1. Arrangement of reinforcement in model staircase (Cusens and Kuang 1966).
Note: 1" = 25.4 mm, 1 ft = 0.3048 mm.



△ Siev's method;

⊠ Gould and Taleb's method;

■ Cusens and Kuang's method;

•———— stress on top surface;

•- - - - stress on bottom surface;

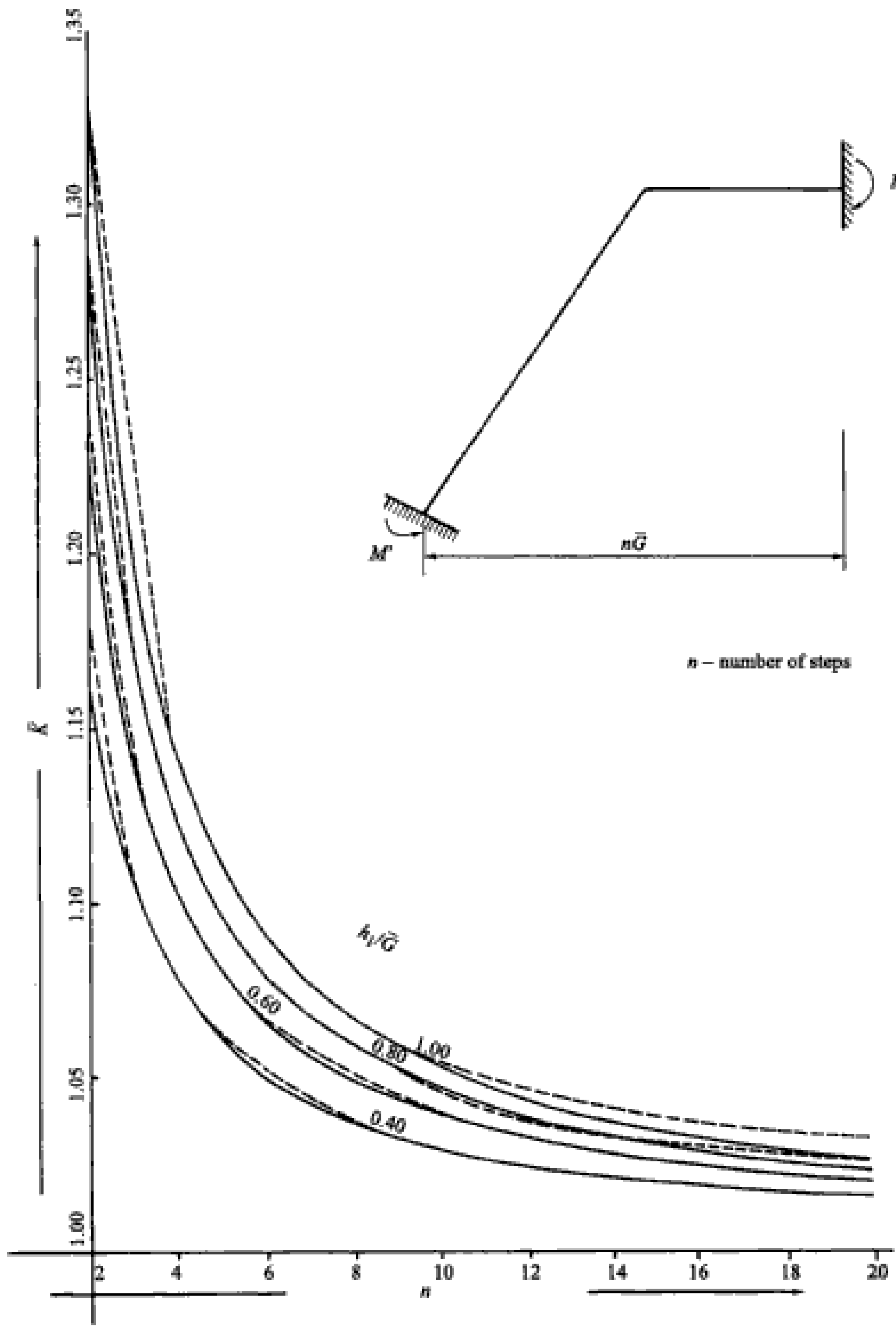
× tension;

– compression;

$\bar{\sigma}$ stress;

w, *q* uniform load;

Figure 4.2. Free-standing stairs – a comparative study of stresses.



- - - - - - - F. E. Plate bending (Bangash);

———— Liebenberg 1956, DIN 1055, Gould 1963, Sauter 1964, classical Cusen and Kuang 1966, Rajagopalan 1973, Leonardt 1977

Figure 4.3. Plotted values of the variation of $\bar{K}$ with n for slabless stairs.

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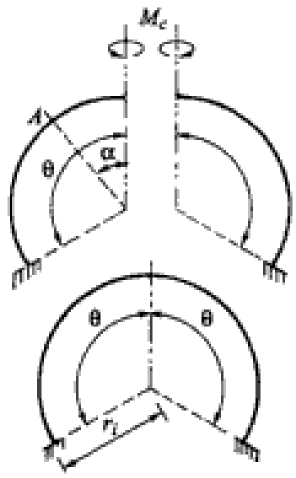

Figure 4.6. Helical stair of Bergman.

Where torsional and direct shear are involved, the stresses are computed as:

$$\sigma = \frac{V}{B} \pm \frac{6M_t}{B^2} \tag{4.6}$$

The straight member of a rectangular section: the maximum torsional shearing stress on a rectangular wide cross section BD_f is given by

$$\tau_{\max} = \tau_{zx} = \frac{M_t}{BD_f^2}\left(3 + 8\frac{D_f}{B}\right) \tag{4.7}$$

where $B \geqslant D_f$.

The maximum torsional stresses paralleling the shorter side is $0.75\tau_{zx}$. The variation or increase of torsional shearing stresses due to curvature is given by

$$\tau_{\max} = \Psi\frac{M_t}{BD_f^2}\left(3 + 1.8\frac{D_f}{B}\right) \tag{4.8}$$

where Ψ is the multiplication factor for stresses.

Figure 4.7 gives the value Ψ which defines the variation.

Bergman's method (Bergman 1956) described above is an approximate method of reducing the helical staircase to that of a horizontal row girder. The structure strength of the helicoidal effect is not considered. Morgan (1960), Holmes (1950) and Scordelis (1960) are based on the longitudinal three-dimensional indetermediate structure of helicoidal shape to the sixth degree. They take advantage of the symmetry and a number of redundants are equal to zero. Holmes (1959) assumes the centre of gravity of the load to act along the centre line of the basic helix and displacements are evaluated using Catigliano's theorem. This differs from Morgan (1960) owing to his location of a centre line of loads parallel, but not coincident with, the centre line of the staircase. A more reasonable approach is that of Scordelis (1960) in which the centre line of the stair is identical to the centre of the stair. Scordelis (1960) also suggests that the eccentricity of the centre line of loads

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CHAPTER 5

Design analysis and structural detailing

5.1 INTRODUCTION

This chapter deals with the design of staircases and other structural components associated with them. A number of stairs have been designed based on the information provided in earlier chapters.

5.2 EVALUATION OF VARIOUS PARAMETERS AND LOADS

5.2.1 *Relation between loads, moments, shears and axial thrusts of inclined and plane projection surfaces*

If the two ends are simply supported (Fig. 5.1)

$$M = \frac{q'(L')^2}{8} = \frac{q\cos^2\alpha(L_3/\cos\alpha)}{8} \tag{5.1}$$

where q and q' are uniform loads on plane projection and on slope, respectively.

The shear force

$$V = \frac{q'L}{2} = q\cos^2\alpha \cdot \frac{L_2}{2\cos\alpha} \tag{5.2}$$

$$\frac{qL_2\cos\alpha}{2} = \overline{V}\cos\alpha \tag{5.2a}$$

The axial thrusts N is given by Eq. (5.3)

$$N = \frac{qL_2\sin\alpha}{2} = \overline{V}\sin\alpha \tag{5.3}$$

where

$$\overline{V} = \text{shear} = \frac{qL_2}{2}$$

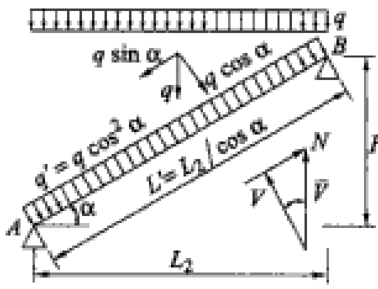

Figure 5.1. Loads on plane and inclined surface.

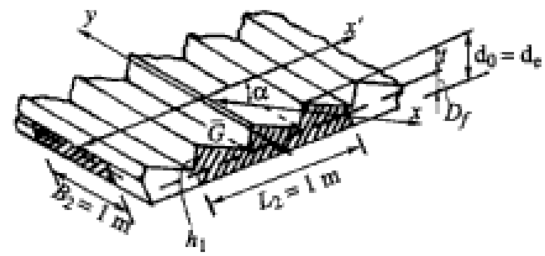

Figure 5.2. Normal stairs.

5.2.2 *Thickness and second moment of areas*

Assuming $L_2 = 1$ m and the width $B = 1$ m for the stair of Figure 5.2 for a normal stair

$$I_x = \frac{t^3}{36} + \frac{D_f}{12} = \frac{t(2t + 3D_f)^2}{36(2 + t/D_f)} = i_1 d_0^3 \tag{5.4}$$

$$I_y = \frac{d_0^2 D_f^2}{6(d_0 + D_f)} = i_2 d_0^3 \tag{5.5}$$

The following table is prepared for t/D_f against values of i_1 and i_2 where $t = h_1 \cos\alpha$ and $d_0 = t + D_f$.

Table 5.1. t/D_f versus values of i_1 and i_2.

t/D_f	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
i_1	0.0278	0.0256	0.0237	0.022	0.0204	0.0190	0.0178	0.0167	0.0157	0.0147	0.0139
i_2	0.0428	0.0418	0.0409	0.0401	0.0394	0.0388	0.0382	0.0377	0.0373	0.0369	0.0365

5.2.3 *Steps and reinforcement*

There are several ways of arranging steps on the main flights of the stairs. The most popular ones are:

a) precast steps in concrete on the staircase flight (Fig. 5.3);
b) steps cast in-situ with the stair case flight (Fig. 5.4);
c) slabless stair with steps doing a dual job (Fig. 5.5).

These figures show, respectively, the reinforcement layouts.

The steps shown in Figure 5.6 are a typical helical stairs. The steps are balanced on a flight. They are doweled into the main flight. The geometry and the analysis are fully dealt with in the text.

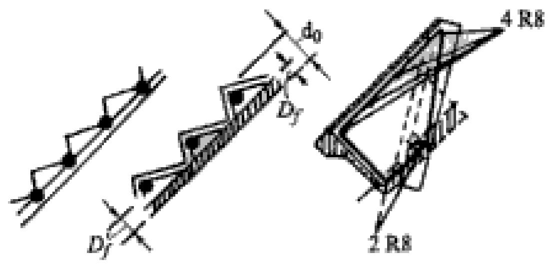

Figure 5.3. Precast concrete stairs. *Note*: $D'_f = D_f +$ depth of embedment.

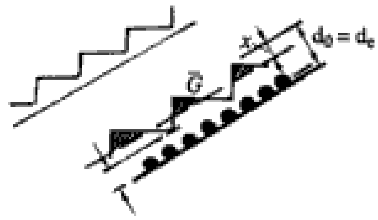

Figure 5.4. Cast in-situ stairs.

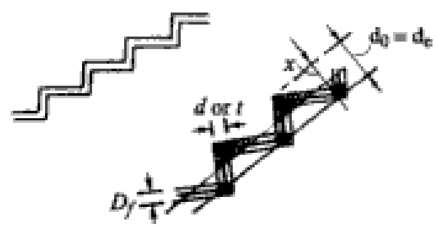

Figure 5.5. Slabless stairs.

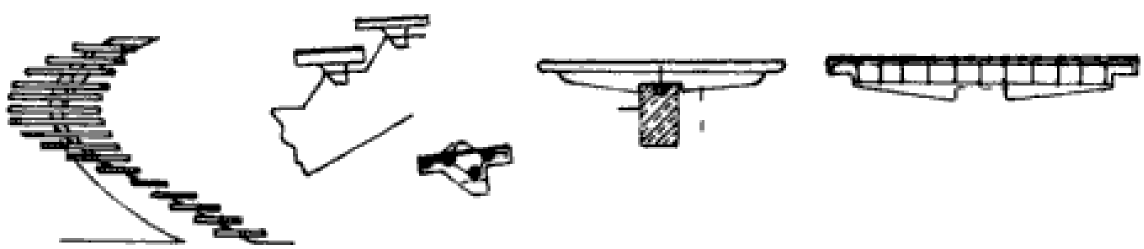

Figure 5.6. Reinforcement details for the steps of a typical helical stairs.

5.3 DESIGN EXAMPLES

Based on the analysis given in the text, a few design examples are given to demonstrate their capabilities. Some numerical examples are already given in Chapters 2 and 3. The same design principles are adopted for them in order to obtain final design drawings.

EXAMPLE 5.1

A typical example is considered for the design of a single flight stair with three different boundary or load conditions. Using the following values and parameters, design this staircase by assuming EI constant as:

Data

Design based on the Limit State Concept

Stair waist	= 175 mm, thick = D_f
Solid finish to treads, risers and the landing	= 40 mm
Plaster to finish	= 0.2 kN/m^2

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at $C = 1.5 \times 24.3$: right $V_{CB} = 36.45$ kN

at $B = -0.5 \times 24.3 = -12.15$ kN

Axial thrust N:

at $A = 0.594 \times 24.3 = 144.342$ kN (comp)

at C, $N_{CA} = -4.02 \times 24.3 = -97.686$ kN (comp) left

at C, $N_{CB} = -4 \times 24.3 = -97.2$ kN (comp) right

Case 2. Moment

$M_C = -0.17 \times 24.3 = -4.131$ kN m

M_{XZ} (when $X = 1$ m) $= 0.41 \times 24.3 = 9.963$ kN m

Case 3. The ends are fixed and the landing is loaded with 1 kN/m. Since the dimensions were different, a re-analysis is carried out below for the height (2.5 m) and a plane projection of the flight (3 m) and the landing (2 m) giving a total horizontal distance of 5 m. The following is the summary of various coefficients, loads, moments etc.

$f_{11} = 1.3, \quad f_{22} = 2.08, \quad f_{33} = 0.67, \quad f_{13} = f_{31} = 0$

$f_{12} = f_{21} = 0.65$

$\delta_{10} = 0, \quad \delta_{20} = 4/3, \quad \delta_{30} = 4/3$

the matrix $[f]_{3\times3}$ is solved for X_1, X_2 and X_3

$X_1 = 0.1914, \quad X_2 = -0.383 \quad \text{and} \quad X_3 = -1.69$

$M = m_0 + m_1X_1 + m_2X_2 + m_3X_3$

$M_A = 0.1914 \times 24.3 = 4.64$ kN m

$M_C = -0.383 \times 24.3 = -9.31$ kN m

$M_B = -1.69 \times 24.3 = -41.07$ kN m

The maximum positive moment in span $CB = -0.45 \times 24.3$
$= 10.94$ kN m.

Stair design based on the British and European Codes. There are three possible cases mentioned which have to be examined. The staircase has to be checked against them. Under Case 1, the maximum moment at $C = -97.2$ kN m; $V_C = 36.45$ kN; $N = 97.686$ kN; the maximum moment at A and $B = 0$; $V_A = 21.87$ kN; $N_A = -144.342$ kN. These values are chosen for the design of this staircase.

Main flight slab:

$$K = \frac{M}{bd^2 f_{cu}} = \frac{97.2 \times 10^6}{1350 \times 149^2 \times 35} \approx 0.0927 < 0.156 = K'$$

$$d = D_f\left(\text{cover} + \frac{1}{2}\text{bar}\right) = 175 - 20 - 6 = 149 \text{ mm}$$

no compression steel is required and the slab thickness is adequate

$$z = d\left[0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right] = 0.88 > 0.95d \text{ OK.}$$

$$A_s = \frac{M}{0.87 f_y z} = \frac{97.2 \times 10^6}{0.87 \times 460 \times 0.88 \times 149}$$

$$= 1852 \text{ mm}^2/1.35 \text{ m width}$$

or $A_{s\,(\text{required})} = 1372$ mm^2/m width

as provided $A_{s\,(\text{provided})}$ T16 bars at 125 mm centres [T16-125],

$[A_{s\,(\text{provided})} = 1608 \text{ mm}^2]$.

The minimum reinforcement for high tensile distribution bars to be provided should be 0.13% of the gross cross-sectional area of the slab.

$A_{s\,\min} = 0.13/100 \times (1000 \times 175) = 227.5\ \text{mm}^2/\text{m run}$

Provide 10 mm high tensile bars (HT) at 300 mm centres ($A_s = 262\ \text{mm}^2/\text{m run}$). First check the bar spacing to satisfy cracking condition ($D_f = 175 > 200$ mm); the clear distance should not exceed the lesser of $3d$ or 750 mm. The value of $3d - 447$ is the maximum clear distance. Both main and distribution steel spacings are within the established limit.

Shear force V:

The ultimate design shear force at C, $V_C = 36.45$ kN $> V_A$ or V_B. This value of shear is considered and the reinforcement designed and checked for 36.45 kN should be maintained throughout.

$$v = \frac{V}{b_v d} = \frac{36.45 \times 10^3}{1350 \times 149} = 0.18\ \text{N/mm}^2$$

$$\frac{100A_s}{b_v d} = \frac{100 \times 1608}{1000 \times 149} = 1.08 > 3\% \quad \text{and} \quad \frac{400}{d} = \frac{400}{149} > 1.0$$

Note: V was computed on the basis of 1350 m width.

The design concrete shear stress v_c is computed from the following equation:

v_c – allowable shear stress $= 0.79[100A_s/b_v d]^{1/3}\ (400/d)^{1/4} f_{pr}$

grade 25 concrete

For grade 35 concrete: $f_{cu} = 35\ \text{N/mm}^2$

$v_c = 0.86208(f_{cu}/25)^{1/3} = 0.965\ \text{N/mm}^2 > 0.18$

No shear reinforcement is necessary.

If the far ends are fixed having the same dimensions and this time the landing is loaded, the moment will be different. Here at the fixed end at B, the top part of the landing slab should be reinforced additionally for a moment of that magnitude. A similar calculation should be carried out for the evaluation of reinforcement.

Figure 5.7 shows that for any or all of the conditions, the staircase design is adequate.

Check for deflection

Span/Depth $= 5.0/0.175 = 28.57 > 26$

$$\frac{M}{bd^2} = \frac{97.2 \times 10^6}{1350 \times 149^2} = 3.2431$$

Modification factor for the tension reinforcement

$$\text{since } f_s = \frac{5}{8} \times 460 \times \frac{1372}{1608} = 245.3\ \text{N/mm}^2$$

$$\text{M. F.} = 0.55 + \frac{477 - 245.3}{120(0.9 + 3.2431)} = 1.016 \leqslant 2.0$$

Allowable span to effective depth for tension reinforcement $= 26 \times 1.016 = 26.4 < 28.57$. At C a beam is placed in order to reduce the span to 3 m. Actual span/depth $= 3000/149 = 20.13 < 26$ the deflection requirement is adequate.

Finite element analysis

Four noded isoparametric elements	= 150.
Two noded bar elements placed on and in the body of the solid elements	= 390.
Factor of safety	= 2.39

EuroCode 2

Based on details given in Appendix 1, the design given in this example is safe.

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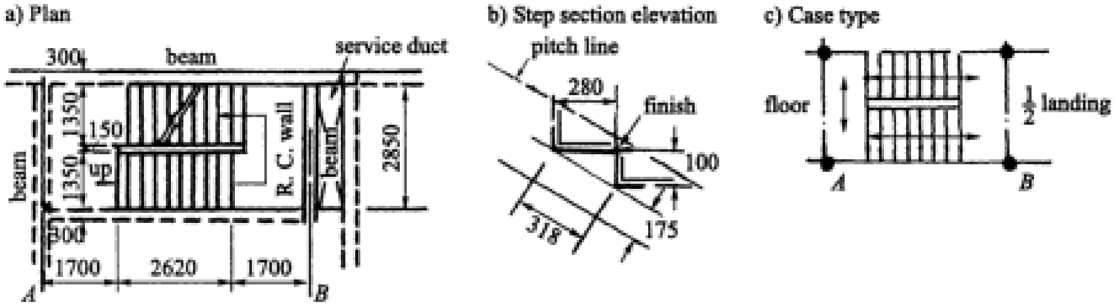

Figure 5.8. Staircases without spine beams.

Total load = 93.3 kN, $R_A = 41.1$ kN and $R_B = 52.2$ kN

Similar work based on the flexibility method has been carried out. The maximum bending moment $M = 69.8$ kN m at 2.52 m from B.

Note: for a span of 1.7 m the load applied = 18.9 kN.

For span of 2.52 m the load applied = 24.3 kN m leaving 850 mm for the end A

$$K = \frac{M}{bd^2 f_{cu}} = \frac{69.8 \times 10^6}{1350 \times 149^2 \times 30} = 0.078 < K' = 0.156$$

$$z = d\left[0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right] = 0.904d < 0.95d$$

$$A_s = \frac{M}{0.87 f_y z} = 1295 \text{ mm}^2/2350 \text{ mm width}$$

$$\text{or} = 960 \text{ mm}^2/\text{m width}$$

Adopt 13T12 bottom equally spaced in 1350 mm width of the stair and 1T10 per step distribution reinforcement. The minimum steel as before T10 at 300 centres (A_s = 262 mm^2). The reinforcement is adequate against cracking.

Perimeter of steel required as U Bar made of mild steel from the R. C. wall is 160 mm,

$$\frac{52.2 \times 10^6}{160 \times 149} = 2.2 \text{ N/mm}^2 \text{ as a value for bending stress is OK.}$$

Hence 7R10 – U Bars from R. C. wall $A_{s\,(\text{provided})} = 553$ mm^2

Landing load = 14 × 3.15 × 1.7 = 75 kN

Two flight = 2 × 41.1 = 82.2 kN = 157.2 kN

Maximum main landing slab moment (span 2850 + 300 = 3150 mm)

width = 1700 mm

$M = 62$ kN m from the flexibility method

area of steel as per width 1700 mm = 1270 mm^2 [11T12 equally spaced in 1700 mm width].

The new reinforcement layout is shown in Figure 5.9.

Finite element analysis

Four nodded isoparametric elements = 150.

Bar elements = 430.

Factor of safety against design = 2.51.

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Case 2. Monolithic with beams

$M = 0.1wL_2 = 20.71$ kN/m

$V_{max} = 24.1$ kN

The reinforcement designed for Case 1 and thickness are sufficient. In Case 2 a rebar arrangement would be necessary to bring about the monolithic state between the flight and the supporting beams.
(B) BS8110 British Code

Design load $= 1.4g_k + 1.6g_k = 1.4 \times 7.944 + 1.6 \times 5.0 = 19.12$ kN/m^2

Case 1.

$M = 0.125 \times 19.12 \times 4^2 = 38.24$ kN m

$$K = \frac{M}{bd^2 f_{cu}} = \frac{38.24 \times 10^6}{1000 \times 172^2 \times 30} = 0.0431 < K' = 0.156$$

No compression steel is needed and the slab thickness is adequate

$z = 0.94964d > 0.95d$ OK.

$$A_s = \frac{M}{0.87 f_y z} = \frac{38.24 \times 10^6}{0.87 \times 250 \times 0.9496 \times 172} = 1077 \text{ mm}^2/\text{m } (A_{s\,(\text{required})})$$

R16-150 $\quad A_{s\,(\text{provided})} = 1340$ mm^2/m

for comparison R16-160 $\quad A_{s\,(\text{provided})} = 1263$ mm^2/m

check for shear $v = \dfrac{V}{b_v d} = \dfrac{1 \times 19.12 \times 10^3 \times 4}{2 \times 1000 \times 172} = 0.22$

$v_c =$ allowable shear stress (25 grade concrete)

$$= 0.79\left[\frac{100A_s}{b_v d}\right]^{1/3} \times \left(\frac{400}{d}\right)^{1/4} = 0.70$$

Grade 30 concrete $v_c = 0.70 \times \left(\dfrac{30}{25}\right)^{1/3} = 0.784 > 0.22$

No shear reinforcement is required.

Case 2. Monolithic beams

The design of the Case 1 is not affected.

EuroCode 2

Based on details given in Appendix 1, the design given in this example is safe. The codes show practically no difference in the final result.

EXAMPLE 5.4

The general arrangement plan of a free-standing staircase of a multi-storey building is shown in Figure 5.11. Using both the Indian Code IS 456, ACI and BS8110, design this staircase which is built around the stairwells as shown in the figure. The following data are adopted.

h_1	= riser height = 150 mm
$\overline{G}$	= going = 250 mm
The stair slab embedded in the wall	= 200 mm
H_1	= effective height = 3 m

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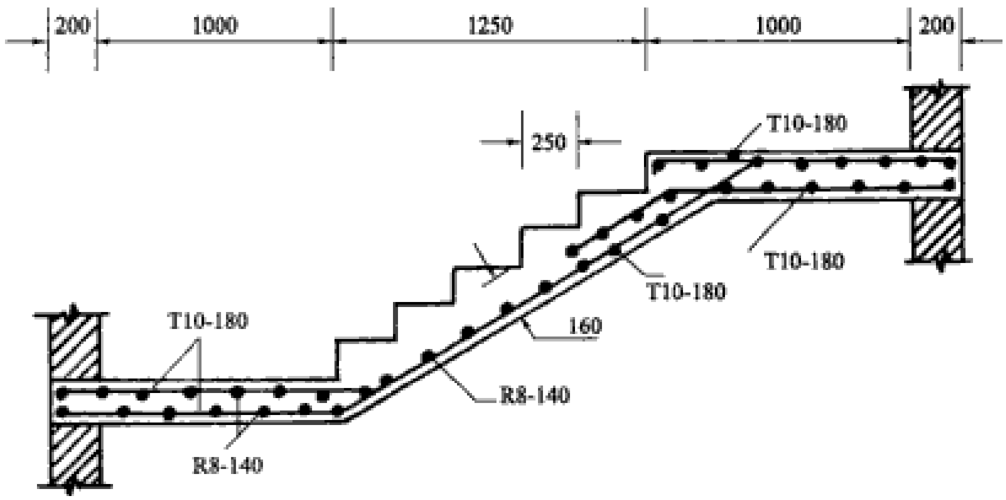

Figure 5.13. Reinforcement details.

$$M_{\max} = 11 \text{ kN m}, \quad V_{\max} = 10.3 \text{ kN}, \quad \frac{L}{30} = \frac{3.45 \times 12}{0.3048 \times 30} = 4.5 \text{ in}$$

$$\text{The weight of the slab} = \frac{4.5}{12} \times 0.15 = 0.056 \text{ kips/ft}^2$$

$$\text{Dead load} = w_D = 0.056 \times 1.4 = 0.079 \text{ kips/ft}^2 \text{ of width}$$

$$\text{Imposed load} = w_L = 0.100 \times 1.7 = 0.170 \text{ kips/ft}^2 \text{ of width}$$

$$M_u = \text{ultimate moment} = (0.079 + 0.170)\left[\frac{3.45}{0.3048}\right]^2 \times \frac{1}{10}$$

$$= 3.190 \text{ kips/ft width}$$

Reinforcement ρ_b equal to about $0.375\rho_b$ or one half the maximum permitted by the ACI Code. In order to have reasonable deflection control Table 5.4 of the code is considered.

$$0.375\rho_b = 0.5 \times 0.0278 = 0.0139 = \rho$$

$$m = \frac{f_y}{0.85 f'_c} = \frac{40{,}000}{0.85 \times 3000} = 15.7$$

$$R_u = \rho f_y\left(1 - \frac{1}{2}\rho_m\right) = 0.0139 \times 40{,}000(1 - 0.5 \times 0.0139 \times 15.7)$$

$$= 495 \text{ psi}$$

$$\text{required } d = \left(\frac{M_u}{\phi R_u b}\right)^{1/2} = \left(\frac{3.19 \times 12{,}000}{0.9 \times 459 \times 12}\right)^{1/2} = 2.8 \text{ in}$$

assume #5 bars, $D_{f\,\text{req}} = 2.82 + 0.31 + 0.75 = 3.94 \text{ in} < 4.5 \text{ in}$

provide $d = 4.5 - 0.31 - 0.75 = 3.44$ in

$$\text{shear requirement } V_{u\max} = 1.15\frac{wL_u}{2} = 1.15\frac{0.249\left(\frac{3.45}{0.3048}\right)}{2}$$

$$= 1.62 \text{ kips/ft of width}$$

The design shear strength ϕV_c for a staircase slab without shear reinforcement

$$\phi\left[2f_c' bd\right] = 0.85 \times 2\sqrt{3000} \times 12 \times 3.44 \times \frac{1}{1000} = 3.84 \text{ kips/ft}$$
$$> 1.7 \text{ kips/ft}$$

The stair slab with reinforcement is adequate.
(C) BS8110

Characteristic design load $= 1.4 \times 9.88 = 14.23$ kN/m^2

Characteristic design imposed load $= 1.6 \times 3 = 4.8$ kN/m^2

$= 19.03$ kN/m^2

Characteristic design dead load $= 1.4 \times 7.44 = 10.416$ kN/m^2

Characteristic design imposed load $= 4.8$ kN/m^2

Total load $= 15.216$ kN/m^2

From flexibility method of analysis

$$M_{max} = 40.76 \text{ kN m},\ \ V_{max} \text{ (stair)} = 11.9 \text{ kN} + 8.368 \text{ kN} = 20.269 \text{ kN}$$
$$d = 129 \text{ mm}$$

$$K = \frac{M}{bd^2 f_{cu}} = 40.76 \times 10^6 \{1000 \times 129^2 \times 30\} = 0.0186$$
$$< K' = 0.156$$

$$z = d\left[0.5 + \sqrt{0.25 - \frac{K}{0.9}}\right] = 0.9d \text{ no compression steel required}$$

$$A_s = \frac{M}{0.87 f_y z} = \frac{40.76 \times 10^6}{0.87 \times 460 \times 0.9 \times 129} = 877 \text{ mm}^2/\text{m } (A_{s\,(\text{required})})$$

$A_{s\,(\text{provided})} = [\text{T12-125}]\ \{A_s = 905 \text{ mm}^2/\text{m}\}$

minimum reinforcement are $= 0.13 \times \dfrac{1000}{100} \times 160 = 280$ mm^2/m

$A_{s\,(\text{provided})} = [\text{T10-300}]\ \{A_{s\,(\text{provided})} = 262 \text{ mm}^2/\text{m}\}$

shear $= 2.269$ kN

$v = 2.269 \times 10^3 \times 1000 \times 129 = 0.157$ N/mm^2

$$\frac{100A_s}{bd} = \frac{100 \times 262}{1000 \times 129} = 0.203, \quad v_c = 0.79\left[\frac{100A_s}{b_v d}\right]^{1/3}\left(\frac{400}{d}\right)^{1/4}$$
$$= 0.5 \text{ N/mm}^2$$

$b_v = 1000, \quad d = 129, \quad A_s = 905$ mm^2

for 30 grade concrete $v_c = 0.5\left(\dfrac{f_{cu}}{25}\right)^{1/3}$ no shear reinforcement required.

$= 0.56$

Check deflection:

$$f_s = \frac{5}{3} f_y \frac{A_{s\,(\text{required})}}{A_{s\,(\text{provided})}} = 228.24 \text{ N/mm}^2$$

$$\frac{M}{bd^2} = \frac{40.76 \times 10^6}{1000 \times 129^2} = 2.449$$

$$\text{modification factor} = 0.55 + \frac{477 - 228.24}{120(0.9 + 2.449)} \approx 1.269 \leqslant 2.0$$

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Concrete:

Density $D_c = \gamma_c = 2400$ kg/m^3

$p_{cb} = 7$ N/mm^2

$p_{cc} = 5.3$ N/mm^2

Shear $v = 0.7$ N/mm^2

Avg. bond stress $= 0.83$ N/mm^2

Local bond stress $= 1.25$ N/mm^2

Mild steel:

$p_{st} = 140$ N/mm^2

$p_{sc} = 125$ N/mm^2

Spine beam:

300 mm × 200 mm deep × 3.1 span lower flight

3.5 m span upper flight

Treads:

0.84 × 0.076 m × 1.15 m

Imposed load = 510 kg/m^2

Landing slab = 1 m × 0.75 m × 0.2 m

SOLUTION
Stairs on spine beams using elastic method

Loading:

$$\left.\begin{array}{ll} \text{Treads } 1.15 \times 0.84\text{ m} \times 0.076\text{ m} \times 2400\text{ kg/m}^3 & = 176\text{ kg/m} \\ \text{Spine beam } 1.25 \times 0.2 \times 0.3 \times 2400\text{ kg/m}^3 & = 180\text{ kg/m} \\ \text{Live load } 0.84\text{ m} \times 510\text{ kg/m}^2 & = 428\text{ kg/m} \end{array}\right\}$$

$$= 784\text{ kg/m}$$

In SI units 748 × 9.81/1000 = 7.69 kN/m

Stairs:

$$d = 200 - 40 - \frac{25}{2} = 147.5\text{ mm}$$

$$M_R = \text{resisting moment} = \frac{p_{cb}bd^2}{4} = \frac{7 \times 300 \times 147.5^2 \times 10^6}{4}$$

$$M_{\text{applied}} = \frac{7.69 \times 3.1^2}{8} = 9.25\text{ kN/m}$$

$$\frac{M}{bd^2 p_{cb}} = 0.81, \quad A_s = \frac{9.25 \times 10^6}{140 \times 147.5 \times 0.81} = 553\text{ mm}^2$$

$$V = 7.69 \times \frac{3.1}{2} = 11.9\text{ kN}$$

$$v = \frac{11.9 \times 10^3}{300 \times 147.5 \times 0.81} = 0.333\text{ N/mm}^2 < 0.7\text{ mm}$$

a nominal steel is required

$$\text{local bond stress} = \frac{11.9 \times 10^3}{300 \times 147.5\left(\frac{3\pi}{6}\right)} = 0.66\text{ N/mm}^2$$

< 0.83 satisfactory

steel bars in beams: together with stirrups

2R12 $\{A_s = 226\ \text{mm}^2\}$

3R16 $\{A_s = 603\ \text{mm}^2\} = 829\ \text{mm}^2$ R8-125 centres

Landing:

Cantilever beam in depth of slab 300 mm wide × 200 mm deep

$$R_f = \text{reaction from the upper flight} = 1/2 \times 7.69 \times 3.5 = 13.5\ \text{kN}$$

$$R_s = \text{reaction from cantilever slab} = 1 \times 0.75(0.2) \times 2400\ \text{kg/m}^3 = 360\ \text{kg}$$

$$\text{slab } 1\ \text{m} \times 0.75\ \text{m} \times 0.2\ \text{m} + \text{imposed } 1 \times 0.75 \times 510 = \frac{383\ \text{kg}}{743\ \text{kg}}\ (\text{total } 7.28\ \text{kN})$$

$$\text{total } R = \frac{7.28 \times 9.81}{1000} = 7.28\ \text{kN}$$

$$M_{max} = 7.28 \times 0.5 + 13.5 \times 0.58 = 11.47\ \text{kN m}$$

$$M_{max} - M_R = 0.06\ \text{kN m}$$

$$A_{sc} = \frac{0.06 \times 10^6}{125(147.5 - 52.5)} = 5.3\ \text{mm}^2$$

$$A_{st} = \frac{11.41 \times 10^6}{140 \times 147.5 \times 0.75} + \frac{0.06 \times 10^6}{140(147.5 - 52.5)} = 741.5\ \text{mm}^2$$

Cantilever slab

$$M_{max} = 7.28 \times \frac{0.75}{2} = 2.73\ \text{kN m}\}$$

Total $A_s = 905\ \text{mm}^2$ $\{4\text{R16 (top)} = A_s = 804\ \text{mm}^2\}$

$\{2\text{R8 (bottom)} = A_s = 101\ \text{mm}^2\}$

$$\frac{M}{bd^2 p_{cb}} = \frac{2.73 \times 10^6}{1000 \times 147.5^2 \times 7} = 0.98$$

$$A_s = \frac{2.73 \times 10^6}{140 \times 147.5 \times 0.98} = 135\ \text{mm}^2 \quad \text{R8-200} \quad \{A_s = 251\}$$

check critical shear for landing column diameter 300 mm
shear force = reaction from upper flight + reaction from lower flight + due to cantilever slab

$$= 13.5\ \text{kN} + 7.69\frac{3.1}{2} + 7.28 \times 2 = 13.5 + 11.9 + 14.56 = 39.69\ \text{kN}$$

$$v = \text{shear stress} = \frac{39.69 \times 1000}{\pi(300 + 200) \times 200} = 0.127\ \text{N/mm}^2 < 0.7\ \text{N/mm}^2\ \text{OK.}$$

$$\text{column area} = \pi\frac{300^2}{4} = 70.7 \times 10^3$$

Loading:

upper and lower flights + slab = 39.69 kN

$$\text{own weight } \frac{2.5\ \text{m} \times 70.7(10)^3}{1000 \times 1000} \times 2400 \times \frac{9.81}{1000} = 4.17\ \text{kN}$$

$$\text{Total} = 44.13\ \text{kN}$$

4T16 bars = $804\ \text{mm}^2$

$$P = p_{cc}A_c + p_{sc}A_{sc} = 5.3(70,700 - 804) + 125 \times 804 = 47.5 \text{ kN}$$
$$> 44.13 \text{ kN}$$

main bars: 4T16 satisfactory, OK.

Stirrups: R8-200 centres

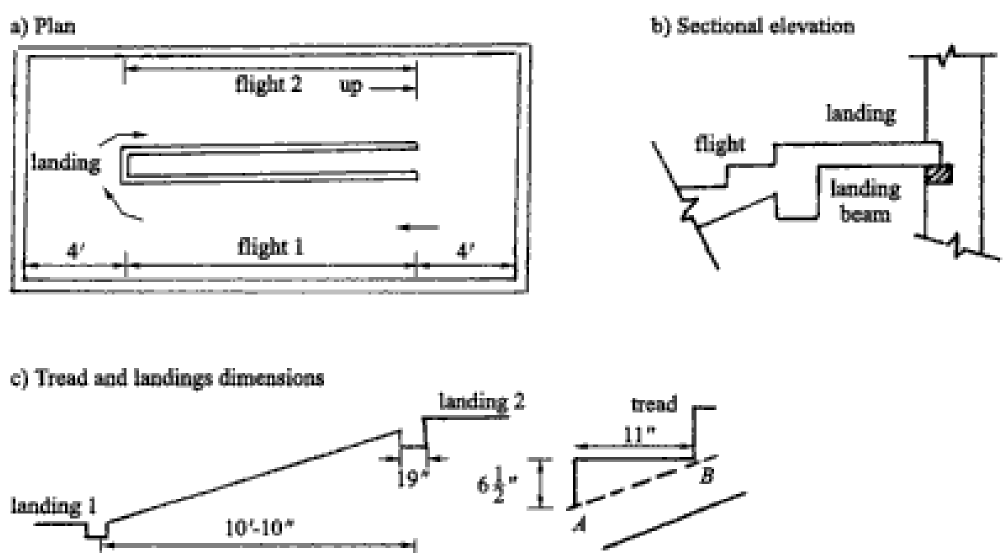
EuroCode 2
Based on details given in Appendix 1, the design given in this example is safe.

EXAMPLE 5.6
In a newly-built bungalow, a RCC free-standing staircase Figure 5.15 is to be designed and constructed in a space specially reserved for it. The internal dimensions of the room are 10 ft × 18 ft (3.048 m × 5.49 m). The height from the first to the second floor is 13 ft (3.96 m). As shown, the staircase should preferably be in two flights. The landing beams are to be constructed and on them the slab rests. Use IS 456 (Indian Standard Institute) and the following data for the design of the staircase:

Steps 4-6″ (1.37 m) wide
Rise = 6.5 inches (165 mm)
Slab thickness = 6″
External walls of the room 13.5 inches (343 mm)
f_c (concrete) = 750 psi (5.171 MN/m^2)
Landing beam (9″) wide and slab 4′ span and 4″ thick
Imposed load = 100 lb/ft^2 (4.788 kN/m^2)
Plan projection of the stairs.
f_t (reinf) = 18,000 psi (124 kN/m^2)
$j = 0.872$

SOLUTION
Staircase in concrete for a bungalow using elastic analysis
Note: 1″ = 25.4 mm; 1 kip = 4.44 kN; 1 ft = 0.3048 m; 1 lbf/in^2 = 6.895 N/m^2.

Figure 5.15. R. C. bungalow staircase.



No. of flight = 2

Height available per flight = 13/2 = 6 ft 6 in

Number of riser per flight = 78″/6.5″ = 12

Number of treads per flight = 12 − 1 = 11

Space available per treads = 18′ − 4′ − 4′ = 10′ = 120″ ≈ 3.074 m

Hence each tread = 121/11 = 11″ wide (280 mm)

(A) Stair slab

beam width given = 9″ (229 mm)

effective span of the slab = 121 + 9 = 130″ (3.302 m)

Loads:

distance $AB = \sqrt{(11^2 + 6.5^2)} = 12.79''$ (325 mm)

load/ft run $= 6 \times 12.79 \times \frac{12}{11} = 83$ lbf/ft (1.211 kN/m)

load due to triangular portion $= \frac{11 \times 6.5}{2} \times \frac{12}{11} 39$ lbf/ft (0.57 kN/m)

dead load $= 83 + 39 = 122$ lbf/ft^2 (5.84 kN/m^2)

total load $= w = 122 + 100 = 222$ lbf/ft^2 (10.63 kN/m^2)

M (using flexibility method) = 39,500 in lbf (4.463 kN m)

when a 12″ width of the R. C. slab is taken

$M = 126d^2 = 39500 \quad \therefore\ d = 5''$ (127 mm)

D_f = total depth of slab $= 5'' + 0.75 + \frac{1}{4}$ (halfdiameter bar)

$= 6''$ (152 mm)

A_s = area of steel $= \frac{39,500}{18,000 \times 0.875 \times 5} = 0.5$ in^2/ft width

$\frac{1}{2}\phi$ bars with pitch $12\frac{\pi}{4}\left(\frac{1}{2}\right)^2/0.5 = 4.70''$ (114 mm) centres

[R12-100 bars $A_{s\,(\text{provided})} = 377$ mm^2/m]

distribution steel 20% of the main steel $= 0.5 \times \frac{20}{100}$

$= 0.10$ in^2 (0.645 cm^2)

pitch $= 12\frac{\pi}{4 \times 16 \times 0.1} = 5.85''$ (147 mm)

use 0.25″ ϕ bars at 5.5″ centres [R8-150 $A_{s\,(\text{provided})} = 335$ mm^2/m]

(B) Landing slab
dead load for 4″ thick slab = 48 lbf/in^2
imposed load $= 4 \times 100 = 400$ lbf/ft width $<$ 850 lbf adopted
imposed load $= \frac{850}{4} = 313$ lbf/ft^2 (15.0 kN/m^2)
w = load on slab $= 213 + 48 = 261$ lbf/ft^2
M at the centre $= \frac{1}{8} \times 261 \times 4^2 \times 12 = 6160$ in lb (0.71 kN m)
d = effective depth $= \sqrt{\frac{M}{126 \times 12}} = 2.05''$ (52 mm)
D_f = total depth $= 2.05 + 0.75 + 0.125 = 2.93'' < 4''$
adopt effective depth $4.0 - 0.75 - 0.125 = 3.13''$

$$A_s = \frac{6260}{1800 \times 3.13 \times 0.872} = 0.13\ \text{in}^2$$

$$0.25''\ \phi \text{ bars pitch} = 12\frac{\pi}{4} \times 0.25^2 \times \frac{1}{0.13} = 4.52''$$

adopt 0.25″ ϕ bars at 4.5″ centres [R8-100 $A_{s\,(\text{provided})} = 503\ \text{mm}^2/\text{m}$]

$$\text{distribution steel } 0.13 \times \frac{20}{100} = 0.026\ \text{in}^2$$

$$\text{pitch} = \frac{12\pi \times 0.25^2}{0.026 \times 4} = 22.74'' > 12''\ (305\ \text{mm})$$

Adopt 0.25″ ϕ bars at 12″ centres [R8-300 $A_{s\,(\text{provided})} = 168\ \text{mm}^2/\text{m}$]
(C) Landing beam (Fig. 5.15(c))
9″ × 18″ section

$$\text{reaction from stair loads} = 222 \times \frac{10.92}{2} = 121\ \text{lbf (1.641 kN m)}$$

Landing loads = one half of the load is borne by the landing beam and the other half is taken by 13.5″ (343 mm) wall.

$$= \frac{1}{2} \times 261 \times 4 = 522\ \text{lbf (2.322 kN)}$$

self weight 18″ × 19″ − 9″ × 4″ of the slab = 126 lbf/ft (1.84 kN)
total load on beam = 1210 + 522 + 126 = 1858 lbf/ft
L_2 = effective span = 10 ft + 9 inches = 10′9″ (3.28 m)
$M = 0.125wL_2^2 = 321 \times 10^3$ in lbf (36.273 kN m)

Since the moment is small, it can be designed as a rectangular beam. In general, for large moments T and L beams should be considered.

$$d = \sqrt{\frac{321 \times 10^3}{126 \times 9}} = 16.82''$$

$D_f = 16.82 + 1''$ cover $+ 0.5''$ for a bar $= 18.32 > 18''$ assumed

9″ × 9″ section (229 mm × 483 mm)

increase in load = 9 lbf/ft (0.132 kN/m)

loading = 1858 + 9 = 1867 lbf/ft (27.3 kN/m)

$$M = 321{,}000 \times \frac{1867}{1858} 323{,}500\ \text{in lbf (36.55 kN m)}$$

$$d = \sqrt{\frac{M}{126 \times 9}} = 16.90'', \quad D_f = 16.90'' + 1'' + 0.5'' = 18.40 < 19''\ (483\ \text{mm})$$

$$d = 19 - 1 - 0.5 = 17.5'', \quad A_s = \frac{323{,}500}{18{,}000 \times 0.872 \times 17.5} = 1.17\ \text{in}^2\ (113\ \text{mm}^2)$$

$$4 - (5/8)''\ \phi \text{ bars}, \quad A_s = \frac{\pi \times 25 \times 4}{4 \times 64} = 1.23\ \text{in}^2/\text{ft}$$

SI comparable 4R16 [$A_{s\,(\text{provided})} = 804\ \text{mm}^2/\text{m}$]

Check for shear:

$$V_{max} = 1867 \times \frac{10}{2} = 9335\ \text{lbf (41.52 kN)}$$

$$v = \text{shear stress} = \frac{V}{bjd} = \frac{9335}{9 \times 0.872 \times 17.50} = 67\ \text{lbf/in}^2$$

$$= 462\ \text{kN/m}^2$$

v_c = allowable shear stress $= 0.1 f_c = 0.1 \times 750 = 75$ lbf/in^2

> 67 lbf/in^2 (462 kN/m^2)

The steel is adequate for shear.
Check for bond stress:

$$v_b = \frac{V}{\sum O jd} = \frac{9335}{4\pi \frac{5}{8} \times 0.872 \times 17.50} = 78 \text{ lbf/in}^2$$

$$= 573 \text{ kN/m}^2$$

Allowable: straightened bar $= 0.1 f_c + 25 = 100$ lbf/in^2

> 78 lbf/in^2 (573 kN/m^2)

hooked ended bar $= 0.2 f_c + 50 = 200$ lbf/in^2

$= 1379$ kN/m^2

the reinforcement is adequate.

EXAMPLE 5.7
The stringer beams of SCS type of 4 m span and spaced at 500 mm centres are used to support a staircase. Using the following data, design the solid stringer beam in timber:

Dead load = 0.6 kN/m^2

Imposed load = 5 kN/m^2 or 9 kN concentrated load

Bending parallel to the grain = 10 N/mm^2

Compression perpendicular to the grain = 2.8 N/mm^2 (without wane)

Shear parallel to the grain = 1.0 N/mm^2

E (modulus of elasticity) = 7100 N/mm^2 minimum

or = 10,700 N/mm^2 (E-mean)

SOLUTION
Timber stringer beam

dead load $= 1.2 + 1.0 = 1.2$ kN (udl)
imposed load $= 5 \times 4 \times 0.5 = 9$ kN concentrated or 10 kN (udl)
long term $= 1.2 + 0 = 1.2$ kN (udl)
medium term $= 1.20 + 10 = 11.2$ kN (udl)
or $= 1.20 + 9 = 10.2$ kN concentrated
greatest stress and deflection: coefficient $K_2 = 1.0$ long term
$= 1.25$ medium term
long term loading $= 1.2 + 1.0 = 1.2$ kN (udl)
medium loading $= 11.2/1.25 = 8.96$ kN (udl)

since the spacing < 610 mm, the load sharing modification factor
$k_8 = 1.1$
max. allowable $= 0.003 \times 4000 = 12$ mm; $E_{mean} = 10,700$ N/mm^2

$$I = \frac{5.0 \times 11,200 \times (4000)^3}{384 \times 10700 \times 12} = 72.7 \times 10^6 \text{ mm}^4$$

A size 75×245 mm would give a bending deflection as

$$\delta_b = \frac{5 \times 11{,}200(4000)^3}{384 \times 10700 \times 91.9 \times 10^6} = 9.49 \text{ mm}$$

$$\delta_v = \text{additional deflection due to shear} = \frac{3wL}{20Gbd}$$

$$= \frac{3 \times 11{,}200 \times 400}{20 \times 669 \times 75 \times 245} = 0.55 \text{ mm}$$

δ = total deflection = 9.49 + 0.55 = 10.04 mm < 12 mm

The modification factor $K_7 = \left(\frac{300}{245}\right)^{0.11} = 1.0225$

Bending parallel to the grain $= 1.1K_8 \times 1.25K_3 \times 1.0225K_7$
$= 14.01 \text{ N/mm}^2$

Compression perpendicular to the grain $= 2.8 \times 1.1K_8 \times 1.25K_3$
$= 3.85 \text{ N/mm}^2$

shear parallel to the grain $= 1.0 \times 1.1K_8 \times 1.25K_3 = 1.375 \text{ N/mm}^2$

shear at the support $= \frac{3F}{2bd} = \frac{3 \times 5600}{2 \times 75 \times 245} = 0.457 \text{ N/mm}^2$
$< 1.375 \text{ N/mm}^2$ OK.

Bearing stress as the support:
The length bearing is 100 mm at the ends of each stringer.

The bearing stress $= \frac{5600}{100 \times 75 \text{ mm}} = 0.74 \text{ N/mm}^2$
$< 3.85 \text{ N/mm}^2$ OK.

EXAMPLE 5.8: *Explain the space truss theory for concrete subjected to torsion*
An unsymmetrical reinforced rectangular section of a stringer beam supporting a staircase is shown in Figure 5.16. From the analysis, the stringer beam is found to be subjected to a torque of 70 kN m. Show that the stringer beam is safe and the reinforcement is adequate.

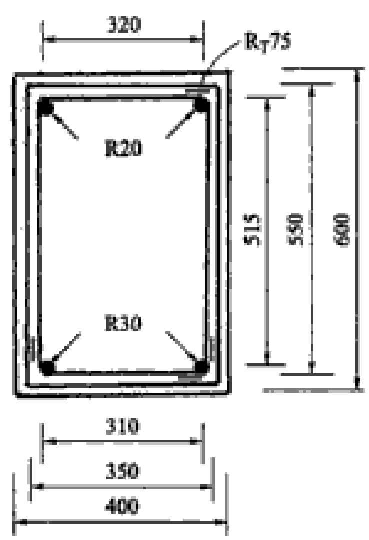


Figure 5.16. Reinforcement in stringer beam.

Use the following data:

$$f_{yt} = \text{yield stresses}$$

$$\left.\begin{aligned} &= \text{for 10 mm } \phi \text{ bar } 250 \text{ N/mm}^2 \\ &= \text{for 20 mm } \phi \text{ bar } 280 \text{ N/mm}^2 \\ &= \text{for 30 mm } \phi \text{ bar } 300 \text{ N/mm}^2 \end{aligned}\right\} \text{ R type}$$

$$b_l = \frac{1}{2}(320 + 310) = 315 \text{ mm}, \quad d_l = 515 \text{ mm}$$

$$b_s = 350 \text{ mm}, \quad d_s = 550 \text{ mm}$$

$$u_0 = 2(b_l + d_l) = 1660 \text{ mm}, \quad A_0 = b_l d_l = 162{,}225 \text{ mm}^2$$

$$s = 75 \text{ mm}, \quad f_{lby} = 300 \text{ N/mm}^2, \quad f_{lty} = 250 \text{ N/mm}^2$$

$$\frac{A_s f_{sy}}{s} = 262 \text{ N/mm}^2, \quad f_{sy} = 250 \text{ N/mm}^2$$

SOLUTION

Application of space truss theory to concrete under torsion

(A) Explanation

Close spacing of longitudinal bars on all faces is considered to be superior in resisting torsion. It is also helpful in controlling the width of the torsional cracks. Figure 17(a) shows a rectangular cross-section with longitudinal bars distributed uniformly on all faces. In the space truss theory it is assumed that the concrete core is not effective and that the compression diagonals are due to the concrete shell. The solid rectangular section, therefore, behave like a hollow section. The equivalent hollow section is shown in Figure 17(b). The effective wall thickness of the equivalent hollow section is then computed.

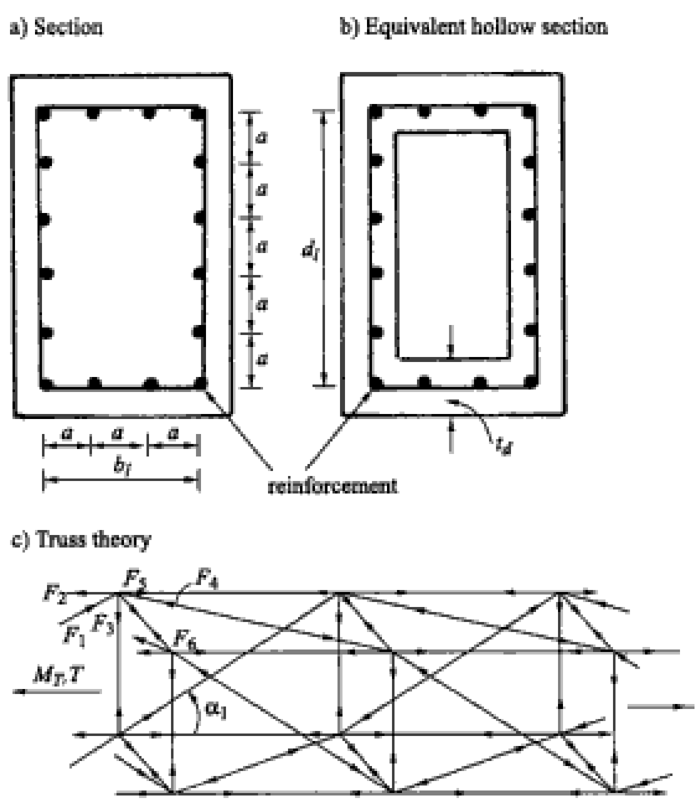


Figure 5.17. Explanatory diagrams for space truss theory.

It is possible for the longitudinal steel to be placed unsymmetrically with reference to the horizontal axis. When the torsion is accompanied by a bending moment, the longitudinal steel near the tension face is greater than that near the compression face. Hence in that case, two types of bars need to be considered as shown in the figure.

If the corner bar is assumed to be equally divided between the two adjacent faces, the total area of the longitudinal steel in the top, bottom and vertical plane trusses (Fig. 17(c) for space truss) is 314 mm^2, 706.5 mm^2 and 510.25 mm^2, respectively. The corresponding field loads are 87.92 kN, 211.95 kN and 149.935 kN, respectively.

(B) Plane truss at top face

$$T_{ul} = 2A_0\sqrt{\frac{a_{lt} f_{lty} \times a_s f_{sy}}{l_t \times s}} = 87.7 \text{ kN m}$$

where T_{ul} = the ultimate torque based on the torsional strength of the top.
If α_1, is the angle of the compression diagram.

$$\cot^2 \alpha_1 = \frac{A_1 F_1}{u_0} \times \frac{s}{a_s f_y} = \frac{87{,}920}{315 \times 262} = 1.0653$$

$$\alpha_1 = 44.1°$$

Steel stresses due to torque of 70 kN m

(C) Distributed longitudinal bar

$$T_{uv} = 2A_0\sqrt{\frac{a_{ly} f_{lvy}(a_s f_{sy})}{l_v \times s}} = 223.3 \text{ N/mm}^2 \text{ OK.}$$

four corner bars

$$T_u = 2A_0\left[\frac{a_{lt} f_{lty} + a_{lb} f_{lby}}{2d_1} \times \frac{a_s f_{sy}}{s}\right]^{1/2} = 199.4 \text{ N/mm}^2 \text{ OK.}$$

(D) Plane truss at the bottom face

$$T_{ub} = 2A_0\left[\frac{d_{lb} f_{lby}}{l_b} \times \frac{a_s f_{sy}}{s}\right]^{1/2} = 136.2 \text{ kN m}$$

$$\cot^2 \alpha_1 = \left[\frac{211{,}950}{315}\right]\frac{1}{262} \quad \text{or } \alpha = 32°$$

$$f_{lb} = \frac{70 \times 10^6 \times 1.6025}{2 \times 162{,}225 \times \frac{706.5}{315}} = 154.1 \text{ N/mm}^2 \text{ OK.}$$

$$f_s \text{ (bottom horizontal leg)} = \frac{70 \times 10^6}{2 \times 162{,}225\left(\frac{780}{75}\right) \times 1.6025}$$

$$= 128.4 \text{ N/mm}^2 \text{ OK.}$$

(E) Plane truss at the vertical face

$$T_{ub} = 2A_0\left[\frac{a_{lt} f_{lty}}{b_l} \times \frac{a_s f_{sy}}{s}\right]^{1/2}$$

$$T_{ub} = T_{vt} = T_u$$

$$\cot^2 \alpha_1 = \left(\frac{149{,}935}{515}\right) \times \frac{1}{262} \quad \text{or } \alpha_1 = 43.5°$$

$$f_s \text{ (vertical leg)} = \frac{70 \times 10^6}{2 \times 162{,}225 \times \frac{78.6}{75} \times 1.0541} = 195.3 \text{ N/mm}^2 \text{ OK.}$$

(F) Truss theory

If the same longitudinal reinforcement is distributed along the four faces, the ultimate torque is given by the equation,

$$T_u = \left[2\frac{a_{lt} f_{lty} + a_{lb} f_{lby}}{u_o} \times \frac{a_s f_{sy}}{s}\right]^{1/2} = 99.8 \text{ kN m}$$

Hence a reduction factor of 87.7/99.8 = 0.879 is introduced due to the asymmetry of the longitudinal steel.

Corner bar: division between the longer and the shorter faces in the ratio d_l/b_l

	Top	Bottom	Vertical
Area (mm^2/mm)	0.4627	1.7025	1.2295
Yield load (N/mm)	129.6	510.75	361.3

The ultimate torque at the top (129.6 N/mm) as a lowest value, the value of T_u = 76.4 kN and the reduction factor is 76.4/99.8 = 0.766.

The stringer can take the ultimate torque of 70 kN m on the basis of allowable stresses and other parameters given in the data. Stresses from axial effects and pure bending as described in previous problems should be algebraically added to these stresses from torsional effects.

EXAMPLE 5.9

The steel stringer is laterally restrained at the ends and at points where the reactions from the stair panels occur as shown in Figure 5.18. Using the following data, check the stringer for bending, buckling, shear and deflection:

Data

Point loads: at $B = 3$ kN

at $C = 2$ kN

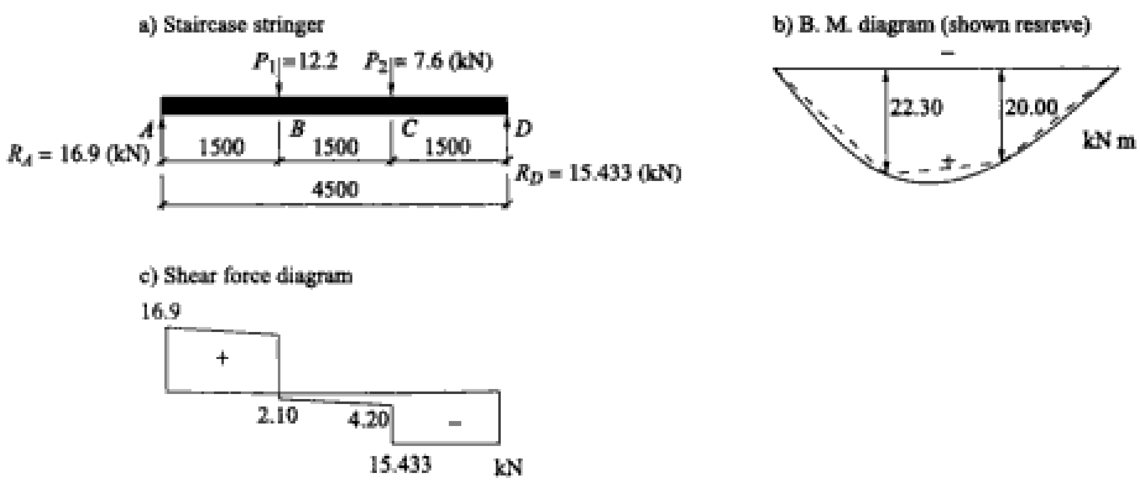
Self weight = 1 kN/m

$E_s = 200$ GN/m^2

$\gamma_L = 1.6$

$\gamma_g = 1.4$

Figure 5.18. Stringer.



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Buckling resistance

M_B = buckling resistance moment = $p_b S_x$

Ref: BS5950, Table 5.5 for $p_y = 275$ N/mm^2

$\lambda_{LT} = nuv\lambda = 23.47$

u = buckling parameter = 0.846

x = torsional index = 14.10

$n = 1.0$

$p_y = 275$ N/mm^2

$\lambda = 28.9$

p_b (Table 5.5) = 275 N/mm^2 $\quad v = 0.96, \quad \dfrac{\lambda}{x} = 2.05$

$M_b = 275 \times 652 \times 10^3 \times 10^{-6} = 179.3$ kN m > 20.739 kN m

$\overline{M} < M_b$ the selection is adequate for the lateral torsional buckling resistance.

Deflection

The imposed load is without safety factors

$$p = \frac{18 \times 12.9}{4.5} = 21.06 \text{ kN}, \quad R_A = 4.3 \text{ kN}$$

$$R_B = 3.7 \text{ kN}$$

actual deflection δ_a

$$\delta_a = \frac{21.06 \times 10^3 \times 4500^3}{200 \times 10^3 \times 6090 \times 10^4} = 2.05 \text{ mm}$$

$$\text{deflection limit } \delta_p = \frac{\text{span}}{360} = \frac{4500}{360} = 12.5$$

$\delta_a < \delta_p$ the stringer is adequate for deflection.

Finite element analysis

Solid elements = 59.

Analysis steps = 15.

Factor of safety = 3.16.

EXAMPLE 5.10

Two stringers 17″ × 14″ support a 3″ slab of the flight spanning 20 ft between the ground floor and the first floor. The stringers are at 6 ft centre to centre. From the flexibility analysis ($L_1 = 20$ ft), the maximum positive and the negative moments are $0.062wL_1^2$ and $0.091wL_1^2$, respectively. Using the following data and the ACI 318.1M89/3.18RM-89 (Revised 1992) Code, design the reinforcement for the stair:

Imposed load = 3 kip/ft

Dead load = 1 kip/ft

Partial } δ_L = 1.7

Safety factor } δ_d = 1.4

$f'_c = 4000$ psi

$f_y = 60$ Ksi

w = uniform load

The stringer is assumed to be cast in-situ with the 3″ slab of the flight. Assume that the torsional effects are included in given moments.

SOLUTION

Stair design using the ACI Code 318-89 (Revised 1992) maximum depth of beam 20″

Note: 1 in = 25.4 mm; 1 kip = 4.448 kN; 1 ft = 0.3048; 1 lbf/in^2 = 6.895 N/m^2.

$M_{LL} = 3 \times 1.7(20)^2 \times 12 \times 0.0625 \quad = 1530$

$M_{DL} = 1.0 \times 1.4(20)^2 \times 12 \times 0.0625 = 420$ positive moment

Total = 1950 in kip

$M_{LL} = 3 \times 1.7(20)^2 \times 12 \times 0.091 \quad = -2225$ negative moment

$M_{DL} = 1.0 \times 1.4(20)^2 \times 12 \times 0.091 = -610$

Total = −2835 in kip

Since it is cast in-situ, flange width $= \frac{1}{4} \times \text{span} = 60''$

d = effective depth = 20 − 2.4 = 17.6″

$$k_u = \frac{M_u}{\phi b d^2} = \frac{195 \times 10^4}{0.90 \times 60(17.6)^2} = 117$$

k_u = flexural strength coefficient

Negative moment with stem width = 14″ and d = 17.6″

d = effective depth = 20 − 2.4 = 17.6″

$$k_u = \frac{M_u}{\phi b d^2} = \frac{283.5 \times 10^4}{0.9 \times 14(17.6)^2} = 727$$

Ref: ACI 318-89 (Revised 1992)

k_u (positive) = 117

$f'_c = 4000$ psi

$$\rho_{\min} = \frac{200}{60{,}000} = 0.0033$$

$\rho_{\text{actual}} = 0.0028 < 0.0033$

$A_s = 0.0033 \times 17.6 \times 60 = 3.52$ in^2

$A_{s\,(\text{provided})}$ = 2#7 1.20 in^2

2#10 2.54 in^2 Total $A_{s\,(\text{provided})} = 3.74$ in^2

Checked for shear and deflection. The stair has adequate provisions.
Finite element analysis

Solid elements = 118.

Steps for the analysis = 15.

Factor of safety = 3.31.

EXAMPLE 5.11

Compute moment, torsion and shear for a helical staircase using the Bergman approximate method and also, using the following data, design the reinforcement for the stair.

r = inner radius of slab of the flight

B = width of slab of the flight = 5 ft

D_f = average normal thickness of slab of the flight = 8.5″

2θ = total angle subtended = 130°

r_i = central radius = 7.5 ft

live load = 100 lb/ft^2 on horizontal projection

dead load including selfweight 175 lb/ft^2 on horizontal projection stair height = 12 ft

$\bar{k} = 0.65$

SOLUTION

Helical staircase – Bergman's method, ACI design method

Note: 1″ = 25.4 mm; 1 kip = 4.448 kN; 1 ft = 0.3048; 1 lbf/in^2 = 6.895 N/m^2.

w = total load/ft = 5 × 275 = 1375 lbf/ft

$$\frac{B}{D_f} = \text{ratio of the flight width to thickness} = \frac{5 \times 12}{8.5} = 7.06$$

$\overline{K} = 0.65, \quad \theta = 65°$

$$U = \frac{2(0.6+1)\sin 65° - 2 \times 0.65 \times \dfrac{65}{57.3}\cos 65°}{(0.65+1)\dfrac{65}{57.3} - (0.65-1)\sin 65° \cos 65°} = 1.18$$

$M_C = 1375 \times 7.5^2(1.18 - 1) = 13{,}922$ ft lbf

$$\alpha \text{ at support} = \theta = 65° = \frac{65}{57.3} \text{ or } 1.1344 \text{ radians}$$

$$\sin\theta = 0.9063078, \qquad \cos\theta = 0.4226183$$

$$M_{support} = wr_i^2(U\cos\alpha - 1) = 1375 \times 7.5^2(1.18 \times 0.4226183 - 1)$$
$$= -38{,}773 \text{ ft lbf}$$

$$T = M_{t(support)} = wr_i^2(U\sin\alpha - \alpha) = 1375 \times 7.5^2 \times (1.18 \times 0.9063078 - 1.1344)$$
$$= -5024 \text{ ft lbf}$$

$V_{support} = wr_i^{\alpha} = 1375 \times 7.5 \times 1.1344 = 11{,}698.5$ lbf

Design of the staircase

In order to distribute reinforcement correctly, similar values of M, M_t and V can be computed at Figure 5.19(a) for various values of α. Here the ACI Code of practice (ACI 1994) is adopted. Figure 5.19(b) shows various diagrams which take into account bending, torsion and shear. Design calculations are similar to the ones given in earlier problems. The final reinforcement details are shown in Figure 5.19(c).

EXAMPLE 5.12

A helical stair beam is subjected to pure torsion and has the cross sectional dimensions shown in Figure 5.20. Check that the reinforcement given is adequate for the following conditions:

a) torsional cracking resistance;
b) torsional stiffness prior to cracking;
c) the factored torsional resistance of the section;
d) torsion stiffness after cracking.

Data

$f_c = 30$ MPa

λ = torsional factor = 1.0

ϕ_c = torsional resisting factor = 0.6

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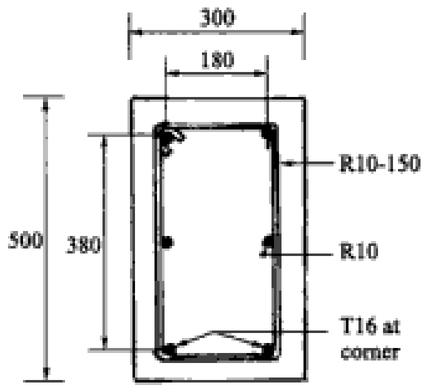

Figure 5.20. A helical stair beam.

$$G = \frac{1}{2}E_c = 5000\sqrt{30} \times \frac{1}{2} = 13.70 \times 10^3 \text{ MPa}$$

$$GC_{gross} = 38.3 \times 10^{12} \text{ N/mm}^2$$

Factored torsional resistance T_r

$$T_r = \frac{2A_o A_t \phi_s f_y}{s} = 29.28 \text{ kN m}$$

A_o = shear flow path

P_b = perimeter of A_{ob}

$A_t = A_s$ = area of tranverse reinforcement

Check

a) longitudinal area of transverse reinforcement

$$A_{lt} = \frac{A_s P_b}{s} = \frac{79 \times 1224}{150} \approx 645 \text{ mm}^2$$

area provided $= 4 \times 201 + 2 \times 79 = 962$ mm^2

b) minimum area of transverse reinforcement (BS8110)

$$= \frac{0.24}{100} \text{ of gross section}$$

$$= 0.24\left(\frac{300 \times 500}{100}\right) = 360 \text{ mm}^2 < 962 \text{ mm}^2$$

Note:

b_b = width between stirrups or links centre line

$= 180 + 10 + 16 = 206$ mm

h_b = depth between stirrups or link centre line

$= 380 + 10 + 16 = 406$ mm

area (A_{ob}) enclosed by stirrup centre line

$= 206 \times 406 = 83 \times 10^3$ mm^2

p = perimeter of A_{ob}

$= 2(206 + 406) = 1224$ mm

A_0 = shear flow path $= 0.85 A_{ob} = 71.09 \times 10^3$ mm^2

A_t or A_s = area of transverse reinforcement = 79 mm^2

Based on CSA Code

$$A_{v\,min} = \frac{0.35 \times 300 \times 150}{460} = 34.24 \text{ mm}^2$$

$$A_{v\,prov} = 2 \times 79 = 158 \text{ mm}^2$$

c) adequacy of section dimensions

$$0.25\lambda\phi_c f_c' = 0.25 \times 1 \times 0.6 \times 30 = 4.5 \text{ N/mm}^2$$

$$\frac{T_r P_b}{A_{ob}^2} = \frac{29.28 \times 10^6 \times 1224}{(83 \times 10^3)^2} = 5.2 \text{ N/mm}^2 > 4.5 \text{ N/mm}^2 \text{ not OK.}$$

The nominal shear stress is excessive. Hence the nominal shear stress caused by the diagonal compression failure in the concrete controls the design

$$T_r \leqslant 0.25\lambda\phi_c f_c' \frac{A_{ob}^2}{P_b} \times 10^{-6} \leqslant 25.33 \text{ kN m}$$

$T_r = 25.33$ kN m is the factored torsional resistance

Finite element analysis

20 noded solid elements = 20.

4 noded bar elements in the body of the solid elements = 10.

Factor of safety = 2.59.

EXAMPLE 5.13

An architect drawing shows the basic layout of the helical staircase as shown in Figure 5.21. The staircase has to be designed in reinforced concrete. Using the following additional data, calculate various moments and shears in the staircase and design the reinforcement at various levels.

$\alpha = \phi$ = slope made by the tangent to helix centre line with respect to the horizontal plane = 25°

$r_i,\ R_i$ = the radius to the inside of the stair = 0.9144 m

β = total arc subtended by helix = 240°

B = width of stair = 1.22 m

$r = R_0$ = radius to the external side of the stair = 2.134 m

$D_f = h$ = minimum thickness of flight = 150 mm or 100 mm

$q_L = w_L$ = superimposed load = 2.873 kN/m²

$D_l,\ \gamma$ = density of concrete = 23.4 kN/m³

f_{cu} = concrete cube strength = 30 N/mm²;

f_y = yield strength of bars = 250 N/mm² or 460 N/mm²

SOLUTION

Helical R. C. staircase – Morgan's method

$R_1 = r_L$ = radius to the centre line of load

$$= \frac{2}{3}\left[\frac{R_0^3 - R_i^3}{R_0^2 - R_i^2}\right] = 1.603 \text{ m}$$

$$R_2 = \frac{1}{2}(2.134 + 0.9144) \approx 1.524 \text{ m}$$

$$\frac{R_1}{R_2} = 1.05, \quad \frac{B}{D_f} = \frac{1.22}{0.100} = 12.2 \text{ m}$$

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M_{rf} = vertical moment about the horizontal axis

$$= M_0 \cos\theta + HR_2\theta \tan\phi \sin\theta - wR_1^2(1 - \cos\theta)$$

at $\theta = 0$, $\phi = 25°$, $M_{rf} = -2.727$ kN m

at $\theta = 120°$, $\phi = 25°$, $M_{rf} = 27.30$ kN m

$M_t = T_f$ = twisting moment

$$= (M_0 \sin\theta - HR_2\theta \cos\theta \tan\phi + wR_1^2 \sin\theta - wR_1R_2\theta)\cos\phi + HR_2 \sin\theta \sin\phi = 7.489 \text{ kN m}$$

M_{nf} = lateral moment

$$= M_0 \sin\theta \sin\phi - HR_2\theta \tan\phi \cos\theta \sin\phi - HR_2 \sin\theta \cos\phi + (wR_1^2 \sin\theta - wR_1R_2\theta)\sin\phi = -37.683 \text{ kN m}$$

Note: for M_t (T_f) and M_{nf} $\theta = 120°$, $\phi = 25°$

P_{nf} = thrust $= -H \sin\theta \cos\phi - wR_1\theta \sin\phi = -35.845$ kN

$\theta = 120°$, $\phi = 25°$

V_{nf} = shear force across the waist of the stairs

$$= wR_1\theta \cos\phi - H \sin\theta \sin\phi = 18.934 \text{ kN}, \quad \theta = 120°, \ \phi = 25°$$

V_{hf} = radial horizontal shearing force $= H \cos\theta$

at $\theta = 0$, $V_{hf} = H = 28.272$ kN

at $\theta = 120°$, $V_{hf} = -14.136$ kN

On the basis of these equations and the given parameters, graphs are drawn for various values of M's and V's. They are given in Figures 5.23(a, b).

Typical design calculations

$M = M_v = M_0$ = moment in a tangential direction

$$= 2.727 \times 10^6 \text{ kN m}$$

$d = 100 - 15 - 12 = 65$ m

$$k = \frac{M}{bd^2 f_{cu}} = \frac{2.727 \times 10^6}{1220(65)^2 \times 30} = 0.0176 < K' = 0.516$$

No compression steel is required.

$$z = d\left[0.5 + \sqrt{0.25 - \frac{k}{0.9}}\right] = 0.98d > 0.95d$$

adopt $z = 0.95d \approx 0.97$ mm

$$A_{s\,(\text{required})} = \frac{2.727 \times 10^6}{0.87 f_y z} = 129.26 \text{ mm}^2/1220 \text{ mm} = 106 \text{ mm}^2/\text{m}$$

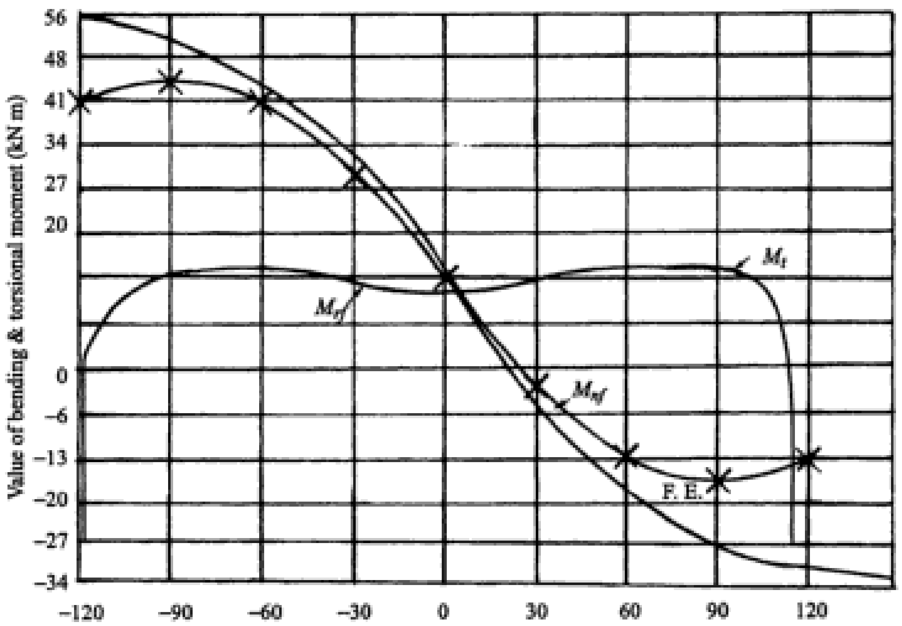
$A_{s\,(\text{provided})}$ = [R12-300] [$A_s = 377$ mm^2/m] or [R10-300]
[$A_s = 262$ mm^2/m]

Minimum area $D/s = 0.24\% \times$ gross sectional area of the flight

$$= 240 \text{ mm}^2/\text{m}$$

[R10-300] [$A_{s\,(\text{provided})} = 262$ mm^2/m] OK.

a) Variation of moments along stair



b) Variation of shearing and thrust along stair

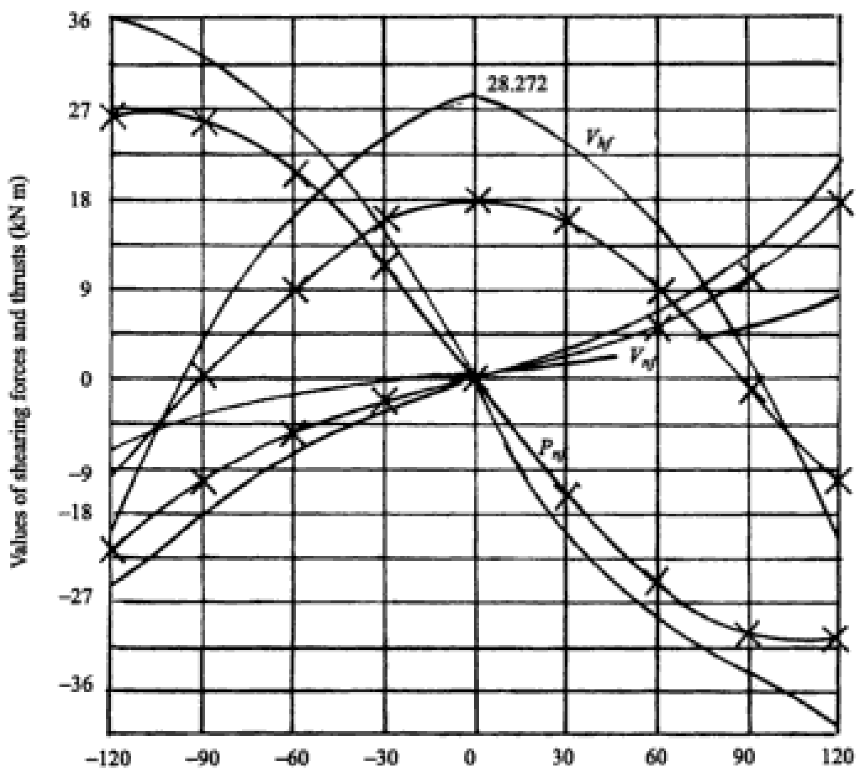


Figure 5.23. Morgan's and finite element methods – A comparative study of shearing and thrust (× – Finite Element).

Cracking due to bending:

In order to ensure that crack widths do not exceed the maximum acceptable limit of 0.3 mm

a) for grade 250 steel D_f or $h > 250$ OK.

b) $\dfrac{100A_s}{bd} = \dfrac{100 \times 262}{1000 \times 65} = 0.4 > 0.3$

a detailed cracking analysis is needed.

The clear distance between bars: the lesser of $3d = 195$ or 750 mm spacing 150 mm between bars is adopted.

Shear:

Ultimate shear $= 18.934 \times 10^3$ N $= V_{nf}$

$$v = \text{the ultimate design shear stress} = \frac{18.934 \times 10^3}{1000(65)} = 0.2913 \text{ N/mm}^2$$

$$\frac{100A_s}{b_v d} = 0.257$$

The allowable design shear stress $= V_c$

$$V_c = 0.79\left[\frac{100A_s}{b_v d}\right]^{1/3}\left(\frac{(400/d)^{1/4}}{\gamma_m}\right) = 0.433 \text{ N/mm}^2$$

$\gamma_m = 1.25$ for grade 25 concrete.

$$\text{for grade 30 concrete } V_c = 0.433\left[\frac{f_{cu}}{25}\right]^{1/3} \approx 0.45 \text{ N/mm}^2$$

since, $v = 0.2913$ N/mm^2 $< v_c < 0.45$ N/mm^2

No shear reinforcement is needed at present under a pure bending condition.

$M_{rf} = 27.30 \times 10^6$ N mm, $\quad K = 0.176 < 0.156$

Increase D_f to 125 mm with $d = 85$ mm

$K = 0.1033 < 0.156$ no compression steel is needed.

No significant change occurs in the calculations for shear or load on the flight

$$z = d\left[0.5 + \sqrt{\left(0.25 - \frac{0.1033}{0.9}\right)}\right] = 0.867d < 0.95d$$

$z = 74$ mm

$$A_{s\,(\text{required})} = \frac{27.3 \times 10^6}{0.87(460) \times 30} \approx 2274 \text{ mm}^2$$

$$A_{s\,(\text{required})} = \frac{1}{1.22} \times 2274 = 1864 \text{ mm}^2/\text{m}$$

T20-150 [$A_{s\,(\text{required})} = 2094$ mm^2/m]

$M_{nf} = 37.68 \times 10^6$ N mm

$d = 1220 - 63 = 1157$ mm

$$K = \frac{37.68 \times 10^6}{150(1157)^2 30} = 0.006255 < K' = 0.156$$

$z = 0.993d > 0.95d$

Take $z = 0.95d = 1099$ mm

$$A_{s\,(\text{required})} = \frac{37.68 \times 10^6}{0.87(460)1099} \times \frac{1}{1.22} = 70.22 \text{ mm}^2/\text{m}$$

Adopt T20-150 as before

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torsional rigidity $= GC$
where G and $C =$ torsional stiffness
values of shear stresses in design for torsion are given by

$$v_{t\,min} = 0.067 f_{cu}^{1/2} \quad \text{but } v_{t\,min} > 0.4\ \text{N/mm}^2$$

$$v_{tu} = 0.8 f_{cu}^{1/2} \quad \text{but } v_{tu} > 5.0\ \text{N/mm}^2$$

A concrete staircase subjected to torsion generally fails as the result of diagonal tension and cracks are formed in a spiral around the slab. The action on each face is similar to the vertical shear in a beam. Reinforcement (of the torsional resistance of all links) crossing the cracks is given by

$$\frac{0.87 f_{yv} A_{sv}}{2}\left(\frac{x_1 y_1}{s_v} + \frac{y_1 x_1}{s_v}\right)$$

thus the torque is given by

$$T = 0.87 f_{yv} A_{sv} \frac{x_1 y_1}{s_v}$$

where, x_1, y_1 are the dimensions of links
A_{sv} – area of two legs of the link; f_{yv} – characteristic strength given on the link.

The crack is assumed to be at 45° (Fig. 5.25)
The expression given in the BS8110: Part 2, clause 2.4.7 is

$$\frac{A_{sv}}{s_v} > \frac{T}{0.8 x_1 y_1 (0.87 f_{yv})}$$

A safety factor of 1/0.8 has been introduced.
Arrangement of reinforcement

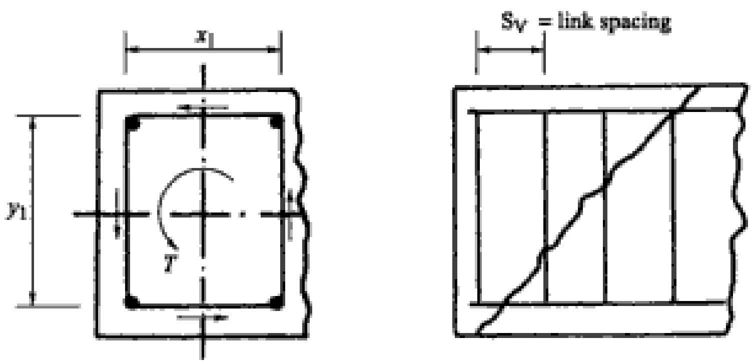

Figure 5.25. Torsional resistance.

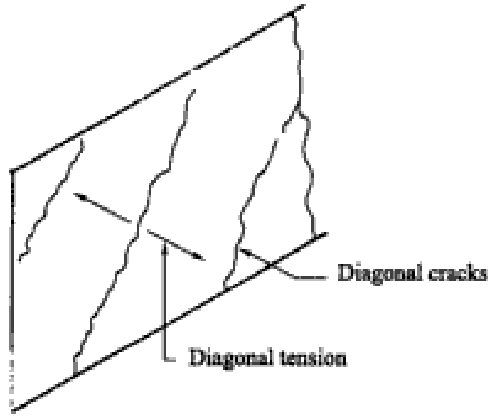

Figure 5.26. Cracks with diagonal tension.

The clear distance between longitudinal bars required to resist torsion should not exceed 300 mm. At present the spacing is 150 mm. Hence 9 bars with a theoretical area of 380/90 = 42 mm^2 per bar are required. For bottom steel

$A_{s\,(\text{required})} = 2274 + 2(42) = 2358\ \text{mm}^2$

$A_{s\,(\text{provided})} = \text{T20 bars} = 2826\ \text{mm}^2$

for top steel 5T20 bars = 1570 mm^2 > 84 mm^2 as top bars in bending were not required.

This arrangement meets the CSA code requirements as well. Figure 5.22 shows the structural details of this type of staircase.

Finite element analysis

Isoparametric 4 Noded

No. solid elements	= 2500.
No. bars matching solid elements (2 noded type)	= 3000.
No. bars in the body of the element	= 1500.
Load types	
No. solution to failure	= 21.
Factor of safety	= 4.15.

EXAMPLE 5.14

A helical horseshoe type staircase is to be designed using the two codes BS8110 and DIN 1045/DIN 1080.

SOLUTION

Design of the horseshoe type staircase

(i) Based on BS8110

The load factors $\gamma_g = 1.4$ and $\gamma_L = 1.6$ are taken into consideration in design of such a staircase. The design calculations are identical to Example 5.13, the final design drawing is shown in Figure 5.27.

(ii) Based on DIN 1045/DIN 1080

The design was carried out by H Vori Winter in Erlauterungen zu DIN 1080 Band: Grundlagen VIII, 144 Seiten ISBN 3-433-00769-1
Published by W Ernst & Sohn 1977

The final design drawing is given in Appendix A2.1.15.

EXAMPLE 5.15

An ellipto-helical R. C. staircase is to be designed. Using the following data, analyze the stair and prepare a useful drawing showing various reinforcement details:

H_1	= staircase height = 2.66 mm
Ellipse in plan	$= \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
h_1	= riser = 190 mm deep 14NO.
$\overline{G}$	= 230 mm = steps width
Waist thickness	= 150 = D_f

Width of the staircase = 0.86 m

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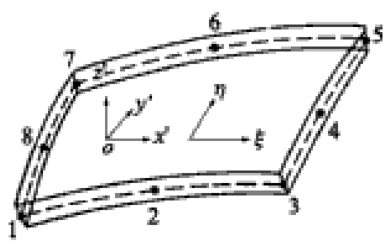
APPENDIX 1

Supporting analyses

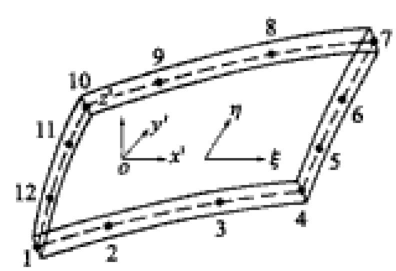
A1.1 SHAPE FUNCTION FOR THE FINITE ELEMENT ANALYSIS

A1.1.1 *Eight-noded membrane element (Bangash 1989)*



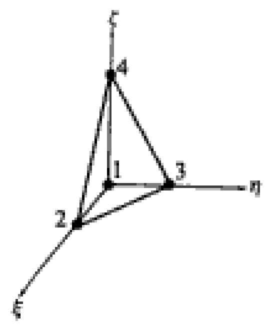
Node i	Shape functions $N_i(\xi, \eta)$	Derivatives $\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$\frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$	$\frac{1}{4}(1-\eta)(2\xi+\eta)$	$\frac{1}{4}(1-\xi)(2\eta+\xi)$
2	$\frac{1}{4}(1-\xi^2)(1-\eta)$	$-\xi(1-\eta)$	$-\frac{1}{2}(1-\xi^2)$
3	$\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$	$\frac{1}{4}(1-\eta)(2\xi-\eta)$	$\frac{1}{4}(1+\xi)(2\eta-\xi)$
4	$\frac{1}{2}(1-\eta^2)(1+\xi)$	$\frac{1}{2}(1-\eta^2)$	$-\eta(1+\xi)$
5	$\frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$	$\frac{1}{4}(1+\eta)(2\xi+\eta)$	$\frac{1}{4}(1+\xi)(2\eta+\xi)$
6	$\frac{1}{2}(1-\xi^2)(1+\eta^2)$	$-\xi(1+\eta)$	$\frac{1}{2}(1-\xi^2)$
7	$\frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$	$\frac{1}{4}(1+\eta)(2\xi-\eta)$	$\frac{1}{4}(1-\xi)(2\eta-\xi)$
8	$\frac{1}{2}(1-\eta^2)(1-\xi)$	$-\frac{1}{2}(1-\eta^2)$	$-\eta(1-\xi)$

A1.1.2 *Twelve noded membrane element (Bangash 1989)*



Node i	Shape functions $N_i(\xi,\eta)$	Derivatives $\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$\frac{9}{32}(1-\xi)(1-\eta)[\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1-\eta)[2\xi-3\xi^2-\eta^2+\frac{10}{9}]$	$\frac{9}{32}(1-\xi)[2\eta-3\eta^2-\xi^2+\frac{10}{9}]$
2	$\frac{9}{32}(1-\xi)(1-\xi^2)(1-\eta)$	$\frac{9}{32}(1-\eta)(3\xi^2-2\xi-1)$	$-\frac{9}{32}(1-\xi)(1-\xi^2)$
3	$\frac{9}{32}(1-\eta)(1-\xi^2)(1+\xi)$	$\frac{9}{32}(1-\eta)(1-2\xi-3\xi^2)$	$-\frac{9}{32}(1-\xi^2)(1+\xi)$
4	$\frac{9}{32}(1+\xi)(1-\eta)[\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1-\eta)[2\xi+3\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1+\xi)[2\eta-3\eta^2-\xi^2-\frac{10}{9}]$
5	$\frac{9}{32}(1+\xi)(1-\eta^2)(1-\eta)$	$\frac{9}{32}(1-\eta^2)(1-\eta)$	$\frac{9}{32}(1+\xi)(3\eta^2-2\eta-1)$
6	$\frac{9}{32}(1+\xi)(1-\eta^2)(1+\eta)$	$\frac{9}{32}(1-\eta^2)(1+\eta)$	$\frac{9}{32}(1+\xi)(1-2\eta-3\eta^2)$
7	$\frac{9}{32}(1+\xi)(1+\eta)[\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1+\eta)[2\xi+3\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1+\xi)[2\eta+3\eta^2+\xi^2-\frac{10}{9}]$
8	$\frac{9}{32}(1+\eta)(1-\xi^2)(1+\xi)$	$\frac{9}{32}(1+\eta)(1-2\xi-3\xi^2)$	$\frac{9}{32}(1-\xi^2)(1+\xi)$
9	$\frac{9}{32}(1+\eta)(1-\xi^2)(1-\xi)$	$\frac{9}{32}(1+\eta)(3\xi^2-3\xi-1)$	$\frac{9}{32}(1-\xi^2)(1-\xi)$
10	$\frac{9}{32}(1-\xi)(1+\eta)[\xi^2+\eta^2-\frac{10}{9}]$	$\frac{9}{32}(1+\eta)[2\xi-3\xi^2-\eta^2+\frac{10}{9}]$	$\frac{9}{32}(1-\xi)[2\eta+3\eta^2-\xi^2-\frac{10}{9}]$
11	$\frac{9}{32}(1-\xi)(1-\eta^2)(1+\eta)$	$-\frac{9}{32}(1+\eta)(1-\eta^2)$	$\frac{9}{32}(1-\xi)(1-2\eta-3\eta^2)$
12	$\frac{9}{32}(1-\xi)(1-\eta^2)(1-\eta)$	$-\frac{9}{32}(1-\eta)(1-\eta^2)$	$\frac{9}{32}(1-\xi)(3\eta^2-2\eta-1)$

A1.1.3 *Shape function tetrahedral element (Bangash 1989)*



Right tetrahedral element

Four-noded
Coordinates:

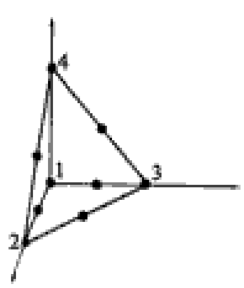
$N_1(\xi,\eta,\zeta) = (1-\xi-\eta-\zeta)$

$N_2(\xi,\eta,\zeta) = +\xi$

$N_3(\xi,\eta,\zeta) = +\eta$

$N_4(\xi,\eta,\zeta) = +\zeta$

Nodal No	ξ_i	η_i	ζ_i
1	0	0	0
2	1	0	0
3	0	1	0
4	0	1	1



Ten-noded
Coordinates:

$$N_1(\xi,\eta,\zeta) = 2(1-\xi-\eta-\zeta)^2 - (1-\xi-\eta-\zeta)$$

$N_2(\xi,\eta,\zeta) = (2\xi-1)\xi$

$N_3(\xi,\eta,\zeta) = (2\eta-1)\eta$

$N_4(\xi,\eta,\zeta) = (2\zeta-1)\zeta$

$N_5(\xi,\eta,\zeta) = 4\xi(1-\xi-\eta-\zeta)$

$N_6(\xi,\eta,\zeta) = 4\xi\eta$

$N_7(\xi,\eta,\zeta) = 4\eta(1-\xi-\eta-\zeta)$

$N_8(\xi,\eta,\zeta) = 4\zeta(1-\xi-\eta-\zeta)$

$N_9(\xi,\eta,\zeta) = 4\xi\zeta)$

$N_{10}(\xi,\eta,\zeta) = 4\eta\zeta$

Nodal No	ξ_i	η_i	ζ_i
1	0	0	0
2	1	0	0
3	0	1	0
4	0	1	1
5	$\frac{1}{2}$	0	0
6	$\frac{1}{2}$	$\frac{1}{2}$	0
7	0	$\frac{1}{2}$	0
8	0	0	$\frac{1}{2}$
9	$\frac{1}{2}$	0	$\frac{1}{2}$
10	0	$\frac{1}{2}$	$\frac{1}{2}$

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A1.2 EUROCODE DATA

Loaded areas		UDL (kN/m^2)	Conc. load (kN)	Ψ_0	Ψ_1	Ψ_2
Category A (domestic and residential activities)	general	2.0	2.0	0.7	0.5	0.3
	stairs	3.0	2.0	0.7	0.5	0.3
	balconies	4.0	2.0	0.7	0.5	0.3
Category B (public buildings, offices, schools, hotels)	general	3.0	2.0	0.7	0.5	0.3
	stairs, balconies	4.0	2.0	0.7	0.5	0.3
Category C (assembly halls, theatres, restaurants, shopping areas)	with fixed seats	4.0	4.0	0.7	0.7	0.6
	other	5.0	4.0	0.7	0.7	0.6
Category D (areas in warehouses, department stores)	general	5.0	7.0	1.0	0.9	0.8

Combination factors (NAD)				
Variable actions		Ψ_0	Ψ_1	Ψ_2
Imposed loads	Dwellings	0.5	0.4	0.2
	Offices and stores	0.7	0.6	0.3
	Parking	0.7	0.7	0.6
Wind loads		0.7	0.2	0
Snow loads		0.7	0.2	0

Load combination	Permanent (γ_G)		Variable (γ_Q)		Wind
	Favourable effect	Unfavourable effect	Favourable effect	Unfavourable effect	
Permanent + variable	1.0	1.35	–	1.5	–
Permanent + wind	1.0	1.35	–	–	1.5
Permanent + variable + wind	1.0	1.35	–	1.35	1.35

γ_G = partial safety factors for permanent actions G
γ_Q = partial safety factors for variable actions Q
Ψ_0 = combination factors for rare load combinations
Ψ_1 = combination factors for frequent load combinations
Ψ_2 = combination factors for quasi-permanent load combinations

$$\sum G_{k,j} + \sum \Psi_{2,i} Q_{k,i}$$

where $i \geqslant 1$; $G_{k,j}$ = characteristic values of permanent actions; $Q_{k,i}$ = characteristic values of variable actions; $\Psi_{2,i}$ = combination factor.

A1.3 TYPICAL EXAMPLE OF A SINGLE STAIRCASE BASED ON EUROCODE 2

The stairs span longitudinally and are set into pockets in the two supporting beams provided. The following data based on Eurocode 2 are provided:

$L_{\text{effective}}$	$= 3$ m
Treads	$= 260$ mm wide
$d_{\text{effective}}$	$= 165$ mm
f_{ck}	$= 30$ N/mm^2
f_k	$= 400$ N/mm^2
$h =$ rise	$= 1.5$ m
Risers	$= 150$ mm
G_k	$= 5.268$ kN/m
E_{cm}	$= 32$ kN/mm^2
f_{yk}	$= 460$ N/mm^2
Waist thickness D_f	$= 200$ mm
Q_k	$= 3$ kN/m
E_s	$= 200$ kN/mm^2
γ_{fQ}	$= 1.5$
ρ_{conc}	$= 24$ kN/m^3

For rare and quasi-permanent combinations of loads we take

$\Psi_0 = 0.7, \quad \Psi_2 = 0.3$

M_U (0.45d upper limit) $= 0.167 f_{ck} b_w d^2$

$$M_{(\text{SLS})} \text{ mid span} = (G_k + \Psi_0 Q_k)\frac{L^2}{8}$$

$$k = \frac{M_U}{b_w d^2 f_{ck}} \leqslant 0.156 = k'$$

Stair slope $= \sqrt{3^2} + \sqrt{1.5^2} = 3.35$ m
$b_w =$ width $= 1$ m of stairs for calculation purposes
Weight of waist and steps $= (0.2 \times 1.0 + 0.26 \times 0.15 \times 1/2) \times 24 = 5.268$ kN/m
Imposed load $= 3.0$ kN/m

Case A: Ultimate load $= 1.35 \times 5.268 + 1.5 \times 3.0 = 11.612$ kN/m (no effective end restraint)

Case B: $M_{(\text{SLS})}$ mid span $= (5.268 + 0.7 \times 3)\dfrac{3^2}{8} = 8.289$ kN m

A1.3.1 *Case A*

$$M_{\text{ultimate}} = \frac{11.612 \times 3^2}{8} = 13.0635 \text{ kN m}$$

$$k = \frac{13.0635 \times 10^6}{1000 \times 165^2 \times 30} = 0.016 < 0.156$$

(no compression steel is provided in the main span)

$z = 0.95d = 0.95 \times 165 = 156.75$ mm

$$A_{s\,(\text{required})} = \frac{13.0635 \times 10^6}{0.87 \times 460 \times 190} = 172 \text{ mm}^2/\text{m}$$

Minimum steel $= \dfrac{0.13}{100} \times 1000 \times 200 = 260$ mm^2/m governs

Provide T10-200 mm centre [$A_s = 393$ mm^2/m]

$$M_u = 0.167 f_{ck} b_w d^2$$
$$= 0.167 \times 30 \times 1000 \times 165^2 = 136.4 \text{ kN m} > 11.612 \text{ kN m applied OK.}$$

A1.3.2 *Case B*

$M_{(\text{SLS})} = 8.289$ kN m

$$\alpha_c = \frac{E_s}{E_{cm}} = \frac{200}{32} = 6.25$$

$$\alpha t = 6\frac{M_{(\text{SLS})}}{b_w D_f^2} = \frac{6 \times 8.289 \times 10^6}{1000 \times (200)^2} = 1.243 \text{ N/mm}^2$$

Steel % (P)

$$p = 0.45 = \frac{50}{\alpha_c} \times \frac{n^2}{1-n^2} \qquad n = 0.21$$

$$1 - \frac{n}{3} = 0.93$$

$$\sigma_s = \text{steel stress} = \frac{M_{(\text{SLS})}}{A_s d\left(1 - \frac{n}{3}\right)} = \frac{8.289 \times 10^6}{393 \times 165 \times 0.93} = 137 \text{ N/mm}^2$$
$$< 0.8 f_{yk} = 368 \text{ N/mm}^2 \text{ OK.}$$

σ_0 = concrete stress

$$= \frac{2M_{(\text{SLS})}}{b_w d^2 n\left(1 - \frac{n}{3}\right)} = \frac{2 \times 8.289 \times 10^6}{1000 \times 165^2 \times 0.21 \times 0.93} = 3.12 \text{ N/mm}^2$$
$$< 0.6 f_{ck} = 0.6 \times 30 = 18 \text{ N/mm}^2 \text{ OK.}$$

A1.3.3 *Deflection*

Eurocode 2 Table 4.14

$$\frac{L}{d} = \frac{3 \times 1000}{165} = 18.2 \text{ related to a steel stress of } 250 \text{ N/mm}^2$$

Corresponding to 400 N/mm^2 = f_{yk} Table 4.14 is multiplied by $\frac{250}{\sigma_s}$, where σ_s = steel stress at that section.

$$\frac{250}{\sigma_s} = \frac{400}{f_{yk}} \times \frac{A_{s\,(\text{required})}}{A_{s\,(\text{provided})}} = \frac{400}{460} \times \frac{172}{393} = 0.3806$$

$$\frac{L}{d} = 0.3806 \times 32 = 12.18$$

For simple span span/depth ratio allowed = 20
Both are less than this value, deflection criteria is satisfied.

A1.3.4 *Cracking*

Check the bar spacing needed to satisfy the cracking Case B for SLS

$D_f = 200$ is at the border line i.e. 200 mm specified.

$$\frac{100A_s}{bd} = \frac{100 \times 393}{1000 \times 165} = 0.238 < 0.3\%$$

for HT steel.
Clear distance between bars must not exceed $3d = 3 \times 165 = 495$ mm or 750 mm.
At present the steel is T10-200 mm centre OK.

A1.4 STAIR STRINGER CONTINUOUS OVER TWO SPANS EITHER SIDE ENV-19

A reinforced concrete stringer supports at each end of the waist slab of the stair. The cross-section of this stringer is T-shaped as shown in Figure A1.4.1. The span of the stringer of this heavy duty stair is 8 m. Intermediate support is provided for the 16 m stringer. The load on each stringer is 97 kN/m. Using the following data, carryout

a) A linear analysis with redistribution (EC-2, 2.5, 3.4.2).

b) A non-linear analysis (EC-2, 2.5, 3.43).

Data

concrete C30/37; $f_{ck} = 30$ N/mm^2; $f_{cd} = 20$ N/mm^2

Reinforcing steel S500 $f_{yk} = 500$ N/mm^2 highly ductile

$f_{ym} = f_{yk} = 500$ N/mm^2; $f_{cm} = 30 + 8 = 38$ N/mm^2

$\gamma_c = 1.00$; $\gamma_s = 1.0$ (without tensioning effects)

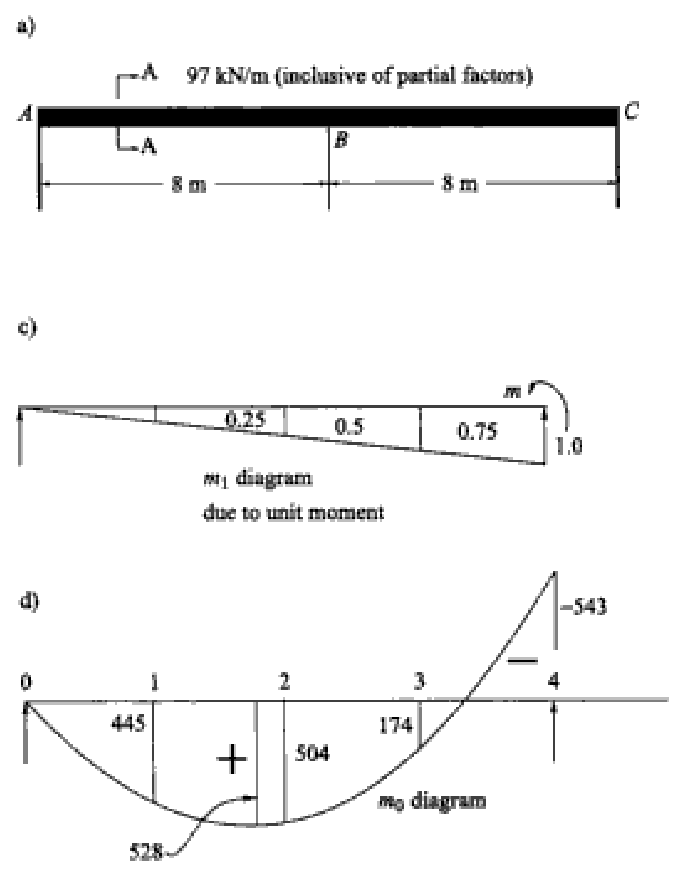
a) Linear analysis

$$M_B = -\frac{97 \times 8^2}{8} = -776 \text{ kN m}$$

δ = redistribution factor = 0.85

Reduced bending moment = $M'_{sd,B} = 0.85(-776) = -660$ kN m

Figure A1.4.1. A continuous stringer for a staircase, placed at both ends.



Check on the cross section design as proposed at support B

$$\mu_{sds} = \frac{660 \times 10^6}{300 \times 750^2 \times 20} = 0.196$$

$$\frac{x}{d} = 0.33 < 0.45$$

$\delta_{\min} = 0.44 + 1.25 \times 0.33 = 0.85$ OK. as above

$V_{sd,A} = V_{sd,C} =$ design value of the applied shear

$$= 97 \times 4.0 - \frac{660}{8} = 306 \text{ kN}$$

$$M_{\max} \text{ (mid span)} = M_{sd1} = M_{sd2} = \frac{V_{sd}^2}{2w} = \frac{306^2}{2 \times 97} = 482.66 \text{ kN m}$$

b) Non-linear analysis

$M_{sd,B}$ over the support B 30% assumed

Check on the rotational capacity

Reduced B. M. at $B = M'_{sd,B} = 0.7(-776) = -543$ kN m

$$\mu_{sds} = \frac{543 \times 10^6}{300 \times 750^2 \times 20} = 0.161$$

$$\frac{x}{d} = 0.263 \text{ (Table 7.1)}$$

Looking at Figure 4.15 of the code $\theta_{pld} = 0.014$
Design value of the applied shear force

$$V_{sd,A} = V_{sd,C} = 97 \times 4 - \frac{543}{8} = 320 \text{ kN}$$

Reinforcement required

$$\text{Maximum value } M_{sd1} = M_{sd2} = \frac{320^2}{2 \times 97} = 528 \text{ kN m}$$

$A_{s\,(\text{required})}$ in spans (Table 7.1 of the code) $= 17$ cm^2 (1700 mm^2/m)

4T10-150 [$A_{s\,(\text{provided})} = 2096$ mm^2]

$A_{s\,(\text{required})}$ at $B = 19$ cm^2 (1900 mm^2/m)

4T10-150 [$A_{s\,(\text{provided})} = 2096$ mm^2]

θ_{required} (Using Simpson's Rule) Flexibility Method

$$= \frac{2\Delta s}{3\sum kM(x)} \cdot \frac{1}{\gamma(x)}$$

$\Delta s =$ internal at the stringer taken to be 2 m

$k =$ coefficient flexibility analysis

$M(x) =$ virtual B

Moment = 1 at support B

$$\frac{1}{\gamma(x)} = \text{curvature at } x \text{ due to applied load.}$$

$$\theta_{\text{required}} = \frac{2(2)}{3} \cdot [0.00529] = 0.007$$

$\theta_{\text{required}} = 0.007 < \theta_{pld} = 0.014$ rotational capacity is not exhausted and is OK.
Hence the size and the reinforcemeat of the stringer is adequate.

Numerical integration

	Points	k	$m(x)$	M	$1/\gamma$	$k(m(x))1/\gamma/m$
	1	2	3	4	5	6
1	0	1	0	0	0	0
2	1	4	0.25	446	0.00283	0.00283
3	2	2	0.50	504	0.00320	0.00320
4	3	4	0.75	174	0.00111	0.00330
5	4	1	1.0	−543	−0.00407	−0.00407
					Total $\sum$	0.00519

A1.5 FIRE PROTECTION ANALYSIS OF LONGITUDINAL STRINGERS BASED ON EUROCODE 3

A1.5.1 *A typical example of steel sections in the stairs*

Determine the thickness of the sprayed plaster protection required to give 90 min fire resistance for a 406 × 178 × 74 UB, Grade S355JR. Use the following data:

$\gamma_{fG} = 1.0, \quad \gamma_{fQ} = 0.8$

Based on ENV 1993-1-2, Clause 4.2.2.2

$$\frac{A_P}{V_i} = 140/\text{m}$$
$$M_D = 532 \text{ kN m}$$
$$M_C = 380 \text{ kN m}$$
$$M_{fi} = 237 \text{ kN m}$$
$$m = 0.89$$
$$p = 20$$
$$\lambda_P = 0.20$$
$$\rho_a = 7850 \text{ kg/m}^3$$
$$\rho_p = 800 \text{ kg/m}^3$$

$$\text{Load ratio } R = \frac{M_{fi}}{M_D} \leqslant m\frac{M_{fi}}{M_c}$$

$$\frac{M_{fi}}{M_D} = \frac{237}{532} = 0.445 \quad \text{also } \mu_{(o)} = \frac{S_{d,f}}{R_{d,f(o)}} = \frac{kS_{d,f}}{R_{d,f}} = 0.70$$

$$\frac{mM_{fi}}{M_C} = \frac{0.89 \times 237}{380} = 0.555$$

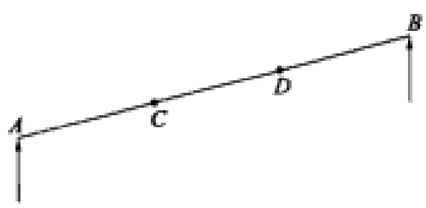


Figure A1.5.1. Fire protection analysis of a stringer.

$R = 0.555$

$$\theta_{a,cr} = 78.38 \ln\left[\left(\frac{1}{0.9674(\mu_o)^{3.833}} - 1\right)^{1/2}\right] + 482 = 624°$$

Effective density $\rho'_p = \rho_p(1 + 0.03p) = 800(1 + 0.03 \times 20) = 1280\ \mathrm{kg/m^3}$

$$I_f = \left[\frac{t_{fid}}{40(\theta_{a,cr} - 140)}\right]^{1.3}$$

$$= \left[\frac{90}{40(624 - 140)}\right]^{1.3} = 9.2 \times 10^{-4}$$

$$\mu = \lambda_p\left(\frac{\rho'_p}{\rho_a}\right)I_f\left(\frac{A_P}{V_i}\right)^2 = 0.2\left(\frac{1280}{7850}\right)^{9.2\times10^{-4}} \times (140)^2 = 0.588$$

$$F_w = \frac{(1 + 4\mu)^{1/2} - 1}{2\mu} = \frac{(1 + 4 \times 0.588)^{1/2} - 1}{2 \times 0.588} = 0.7065$$

d_p = thickness of the spray material for the stringers in the staircase

$$= \lambda_p I_f F_w\left(\frac{A_P}{V_i}\right) = 0.2 \times 9.2 \times 10^{-4} \times 0.7065 \times 140 = 0.0182\ \mathrm{m}$$

or 18 mm spray plaster.

A1.6 A TYPICAL EXAMPLE OF A WOODEN STAIRCASE STRINGER DESIGN BASED ON EUROCODE 5

Figure A1.6.1 shows the stringer of a wooden staircase. Due to landings at A and B and horizontal cross-members for the staircase the reactions at restraints A, B and C are shown. There are axial vertical and horizontal thrusts at these restraints. The stringers are placed parallel to one and other at 0.60 m spacings. The stringers are 38 × 125 sawn timber strength clause C16. They are inclined at 35° which forms the slope of the staircase. Using the data given and some to be taken from the code, check the stringer for both ultimate 1–1 and serviceability limit states. Assume maximum bending occurs in AC.

Data

G_k = 0.425 kN/m on slope

G_{mean} = 8000 kN/mm^2

Q_k = 0.48 kN/m on plan

A-area = 4750 mm^2

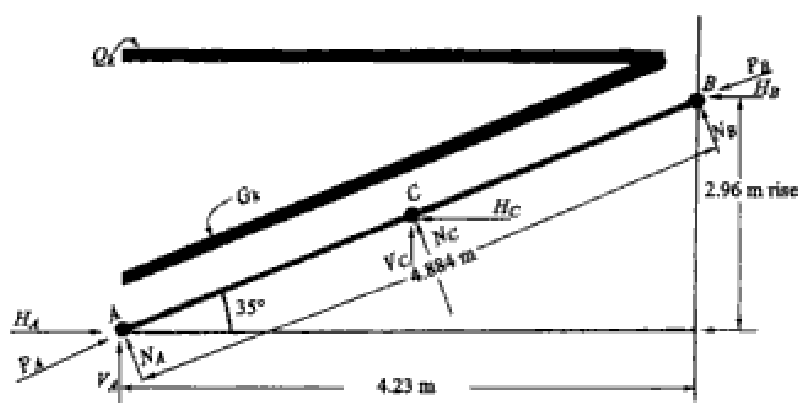


Figure A1.6.1. A wooden stringer.

W_y = section modulus = 98 960 mm^3
I_y = second moment of area = 6 185 000 mm^4
$f_{c,ok}$ = compressive stress = 17 N/mm^2
$E_{0.05}$ = Young's modulus = 5400 kN/mm^2
k_{def} = 1.8 permanent load deflection
= 1.0 short term load deflection
β_c = factor = 0.2
γ_Q = 1.50
γ_G = 1.35
ϕ = form factor = 1.2 (rectangular section)

Permanent load
Normal to stringer = $G_k \cos 35° = 0.348$ kN/m
Parrallel to stringer = $G_k \sin 35° = 0.244$ kN/m

Short term load
Normal to stringer = $Q_k \cos^2 35° = 0.322$ kN/m
Parallel to stringer = $Q_k \cos 35° \sin 30° = 0.226$ kN/m

Permanent loads from building (p) structures

$$N_{AP} = N_{BP} = 0.425 \text{ kN i.e. } \frac{1}{2}(2.442 \times 0.348)$$
$$N_{CP} = 2N_{AP} = 0.850 \text{ kN}$$
$$P_{BP} = 0.425 \cot 35° = 0.607 \text{ kN}$$
$$P_{AP} = 2 \times 2.442 \times 0.244 + 0.425 \cot 35° = 1.799 \text{ kN}$$
$$H_{BP} = \frac{-0.425}{\sin 35°} = -0.741 \text{ kN}$$
$$H_{CP} = -2 \times 0.425 \sin 35° = -0.488 \text{ kN}$$
$$H_{AP} = 1.799 \cos 35° - 0.425 \sin 35° = 1.230 \text{ kN}$$
$$\sum H = 0 \quad \text{OK.}$$
$$V_{CP} = 2 \times 0.425 \cos 35° = 0.696 \text{ kN}$$
$$V_{AP} = 1.799 \sin 35° + 0.425 \cos 35° = 1.38 \text{ kN}$$
$$\sum V = 2.076 \text{ kN}$$
$$\sum V = 2 \times 2.442 \times 0.425 = 2.076 \text{ kN} \quad \text{OK.}$$

Short term loads (q)

$$N_{Aq} = N_{Bq} = \frac{1}{2}(2.442)(0.322) = 0.393 \text{ kN}$$
$$N_{Cq} = 2N_{Aq} = 0.786 \text{ kN}$$
$$P_{Bq} = 0.393 \cot 35° = 0.561 \text{ kN}$$
$$P_{Aq} = 2 \times 2.442 \times 0.226 + 0.393 \cot 35° = 1.665 \text{ kN}$$
$$H_{Bq} = \frac{0.393}{\sin 35°} = -0.685 \text{ kN}$$
$$H_{Cq} = 2 \times 0.393 \sin 35° = -0.451 \text{ kN}$$
$$H_{Aq} = 1.665 \cos 35° - 0.393 \sin 35° = 1.138 \text{ kN}$$
$$\sum H = 0 \quad \text{OK.}$$
$$V_{Cq} = 2 \times 0.393 \cos 35° = 0.644 \text{ kN}$$
$$V_{Aq} = 1.665 \sin 35° + 0.393 \cos 35° = 1.277 \text{ kN}$$
$$\sum V = 1.920 \text{ kN}$$
$$\sum V = 2 \times 2 \times 0.480 = 1.920 \text{ kN} \quad \text{OK.}$$

Ultimate limit state

Design load normal to stringer $= \gamma_G G_K + \gamma_Q Q_K$

$= 1.35(0.348) + 1.5(0.322) = 0.953$ kN/m

Design load parallel to stringer $= 1.35(0.244) + 1.5(0.226)$

$= 0.668$ kN

At B (near top landing) $P_B = 1.35(0.607) + 1.5(0.561) = 1.661$ kN

Shear force $V_d = (0.953)\left(\frac{1}{2} \times 2.442\right) = 1.164$ kN

M_{yd} (bending moment) $= \dfrac{0.953(2.442)^2}{8} = 0.710$ kN m

Axial load at mid point of the bottom part of the stringer

$= 1.5(2.442)(0.668) + 1.661 = 4.108$ kN

τ_d = shear stress $= \dfrac{1.5V_d}{A} = 0.37$ N/m^2

$f_{V,d}$ = shear strength $= \dfrac{K_{15} K_{\text{mod}} F_{V,k}}{\gamma_M} = \dfrac{1.1 \times 0.9 \times 1.8}{1.3}$

$= 1.37$ N/mm^2

$\sigma_{m,y,d}$ = bending stress $= \dfrac{M_{yd}}{W_y} = 7.18$ N/mm^2

$f_{m,y,d}$ = bending strength $= \dfrac{k_h k_{15} k_{\text{mod}} F_{m,k}}{\gamma_M}$

$= \dfrac{1.037 \times 1.1 \times 0.9 \times 16}{1.3} = 12.64$ N/mm^2

i_y = radius of gyration $= \sqrt{\dfrac{I_y}{A}} = 36.08$ mm

λ_y = slenderness ratio $= \dfrac{2442}{36.08} = 67.7$

$\sigma_{c,o,d}$ = axial stress $= \dfrac{4.108 \times 1000}{4750} = 0.87$ N/mm^2

$\sigma_{c,\text{crit},y}$ = buckling stress $= \dfrac{\pi^2 E_{0,05}}{\lambda_y^2} = 11.63$ N/mm^2

$$\lambda_{\text{rel},y} = \sqrt{\frac{F_{c,ok}}{\sigma_{c,\text{crit},y}}} = 1.209$$

$$k_y = 0.5\left(1 + \beta_c(\lambda_{\text{rel},y} - 0.5) + \lambda_{\text{rel},y}^2\right) = 1.32$$

$$k_{c,y} = \frac{1}{k_y + \sqrt{k_y^2 - \lambda_{\text{rel},y}^2}} = 0.560$$

Combined bending and axial stress

$$\frac{\sigma_{c,o,d}}{k_{c,y} F_{c,o,d}} + \frac{\sigma_{m,y,d}}{f_{m,y,d}} \leqslant 1.0$$

$$f_{c,o,d} = \frac{1.1 \times 0.9 \times 17.0}{1.3} = 12.95 \text{ kN/mm}^2$$

hence $\dfrac{0.87}{0.56 \times 12.95} + \dfrac{7.18}{12.64} = 0.69 < 1.0$ OK.

Serviceability limited state-deflection

Permanent service load normal to stringer = 0.348 kN/m

Short term service load normal to stringer = 0.322 kN/m

Flexural deflection due to uniform load

$= \dfrac{5F_{udl}L^4}{384E_{0,\text{mean}}I_y}$ combined for u_1 or u_2

Shear deflection due to uniform load

$$= \frac{\phi F_{udl} L^2}{8 G_{mean} A}$$

F_{udl} for $u_1 = 0.348$

F_{udl} for $u_2 = 0.322$

u_1 = instantaneous permanent deflection = 3.40 mm

u_2 = instantaneous short term load deflection = 3.13 mm

u_{1fin} = final permanent load deflection = $(1 + k_{def})$
= 6.12 mm

u_{2fin} = final short term load deflection = $u_2(1 + k_{def})$
= 3.13 mm

Total deflection = 9.25 mm

Recommended deflections

$$u_{2,inst} = \frac{\text{span}}{300} = \frac{2.442}{300} = 8.14 \text{ mm against } 3.13 \text{ mm}$$

$$u_{net,fin} = \frac{\text{span}}{200} = \frac{2.442}{200} = 12.21 \text{ mm against } 9.25 \text{ mm} \quad \text{OK.}$$

A1.7 COMPUTER PROGRAM SSTRING FOR BM ORDINATES

```
      MASTER SSTRING
C THIS PROGRAM COMPUTES THE REACTIONS: BENDING MOMENT ORDINATES AT
C INTERVALS OF ONE-TENTH OF THE SPAN AND DEFLECTION AT THE CENTRE
C OF SIMPLY SUPPORTED BEAMS WITH TRIANGULAR LOADING
      WRITE(2,1)
1     FORMAT(1H1/////15X,40HREACTIONS, BENDING MOMENT ORDINATES AND ,
     150HDEFLECTION AT THE CENTRE OF SIMPLY SUPPORTED BEAMS//
     248X,23HWITH TRIANGULAR LOADING///////)
    9 READ(1,2) N
2     FORMAT(I3)
      IF(N.EQ.0)GO TO 10
      READ(1,3)S,W,A
3     FORMAT(3F0.0)
C CALCULATE REACTIONS, RA AND RB
      RB=W*(S+A)/6
      RA=W*S/2-RB
      WRITE(2,4)N,S,W,A,RA,RB
4      FORMAT(5X,12HINPUT DATA :///20X,17HBEAM REFERENCE NO,22X,3H = ,
      1I6//20X,4HSPAN,35X,3H = ,F6.3,2H M//20X,21HMAXIMUM LOAD ORDINATE,
      218X,3H = ,F6.3,5H KN/M//20X,32HDISTANCE OF APEX FROM LEFT HAND ,
      33HEND,4X,3H = ,F6.3,2H M/////5X, 9HRESULTS :///20X,
      429HREACTION AT LEFT HAND END, RA,10X,3H = ,F7.3,3H KN//20X,
      530HREACTION AT RIGHT HAND END, RB,9X,3H = ,F7.3,3H KN///
      610X,18HDIST FROM L.H. END 10X,23HBENDING MOMENT ORDINATE/
      715X,8H(METRES),24X,4HKN.M/)
C CALCULATE BENDING MOMENT ORDINATES
      DO 5 I=0,10
      X=I*S/10
      IF(A-0.0)5,12,13
   12 BMX=RB*(S-X)-W*(S-X)**3/(6*S)
      GO TO 5
   13 IF(X-A)6,6,7
```

```
    6 BMX=X*(RA-W*X*X/(6*A))
      GO TO 5
    7 BMX=(S-X)*(RB-W*(S-X)**2/(6*(S-A)))
    5 WRITE(2,8)X,BMX
8     FORMAT(F22.3,F30.3)
C CALCULATE DEFLECTION AT THE CENTRE
      GO TO 9
   10 WRITE(2,11)
11    FORMAT(//////51X,20H**  END  OF  RUN  **)
      STOP
      END

END OF SEGMENT, LENGTH  195, NAME  SSTRING

      FINISH
END OF COMPILATION -   NO ERRORS

S/C SUBFILE:    10 BUCKETS USED

CONSOLIDATED BY XPCK 12B  DATE    18/05/73  TIME 10/55/19

PROGRAM HOPE
EXTENDED DATA (22AM)
COMPACT PROGRAM (DBM)
CORE              4736

                 SEG    SSTRING
                 ENT    FTRAP
                 ENT    FRESET

REACTIONS, BENDING MOMENT ORDINATES AND DEFLECTION AT THE CENTRE OF SIMPLY
   SUPPORTED BEAMS WITH TRIANGULAR LOADING

INPUT DATA :
            BEAM REFERENCE NO                              =     1
            SPAN                                           = 3.000 M
            MAXIMUM LOAD ORDINATE                          = 2.000 KN/M
            DISTANCE OF APEX FROM LEFT HAND END            = 1.500 M

RESULTS :

            REACTION AT LEFT HAND END, RA                 = 1.500 KN
            REACTION AT RIGHT HAND END, RB                 = 1.500 KN

    DIST FROM L.H. END              BENDING MOMENT ORDINATE
         (METRES)                            KN.M

          0.000                   0.000
          0.300                   0.444
          0.600                   0.852
          0.900                   1.188
          1.200                   1.416
          1.500                   1.500
          1.800                   1.416
          2.100                   1.188
          2.400                   0.852
          2.700                   0.444
          3.000                   0.000
```

```
INPUT DATA :
          BEAM REFERENCE NO                             =    2
          SPAN                                          = *300.000 M
          MAXIMUM LOAD ORDINATE                         = 20.000 KN/M
          DISTANCE OF APEX FROM LEFT HAND END           =  0.000 M


RESULTS :

          REACTION AT LEFT HAND END, RA                = *2000.000 KN
          REACTION AT RIGHT HAND END, RB                = *1000.000 KN

   DIST FROM L.H. END        BENDING MOMENT ORDINATE
        (METRES)                      KN.M

         0.000                        0.000
        30.000                    51300.000
        60.000                    86400.000
        90.000                   107100.000
       120.000                   115200.000
       150.000                   112500.000
       180.000                   100800.000
       210.000                    81900.000
       240.000                    57600.000
       270.000                    29700.000
       300.000                        0.000


                                              **  END  OF  RUN  **
```

APPENDIX 2

Structural details for practising engineers

A2.1 DRAWINGS AND STRUCTURAL DETAILS FOR CONCRETE STAIRS

2.1.1 Staircase: Free-standing – Reinforcement details (British practice)
2.1.2 Staircase: Free-standing supported by brickwork or on beams – Reinforcement details (British practice)
2.1.3 Pre cast concrete staircase (Birchwood Products) (British practice)
2.1.4 Pre cast concrete stairs (British practice)
2.1.5 Plans and elevations of R.C. stairs: STEPS (Turkish/European practice)
2.1.6 Typical reinforcement details of stairs and landings in a building: STEPS (Turkish/European practice)
2.1.7 Typical reinforcement details of stairs and landings in a building: WAIST (Turkish/European practice)
2.1.8 Ellipto-helical staircase (Hyder Group UK) (British practice)
2.1.9 Structural details of ellipto-helical staircase (Hyder Group UK) (British practice)
2.1.10 Helical staircase – Elevation and plans (Turkish/European practice)
2.1.11 Helical staircase – Structural details (Turkish/European practice)
2.1.12 Helical staircase – Circular in plan – Reinforcement details (European practice) (Von K. Winter 1977) (Ernst & Sohn)
2.1.13 Helical-cum horseshoe staircase – Reinforcement details (see example) (German practice) Erlauterungen zu DIN 1080, Von K. Winter 1977, Ernst & Sohn (Compliments from Von K. Winter)
2.1.14–2.1.16 Mixed staircase – Straight-cum circular/helical (Ward & Cole, London) (British practice)

A2.2 DRAWINGS AND STRUCTURAL DETAILS FOR STEEL STAIRCASES

2.2.1 Sectional elevation of a steel stairs (Gibbs & Hill, New York) (American practice)
2.2.2 Structural details of stair (Gibbs & Hill, New York) (American practice)
2.2.3 Arch details of stair (Gibbs & Hill, New York) (American practice)
2.2.4 Steel helical staircase – Elevations, plans and structural details (Turkish/European practice)
2.2.5 Sectional elevation, plan and details for free-standing steel stairs (Turkish/European practice)
2.2.6 Typical stringer, step and connection details for steel staircases
2.2.7 Connection details for steel stringers to concrete landings and handrails for steel staircases

A2.3 STRUCTURAL DETAILS IN TIMBER

2.3.1 Typical wooden staircases and their details
2.3.2 Handrails and posts

STAIRCASE ELEVATION

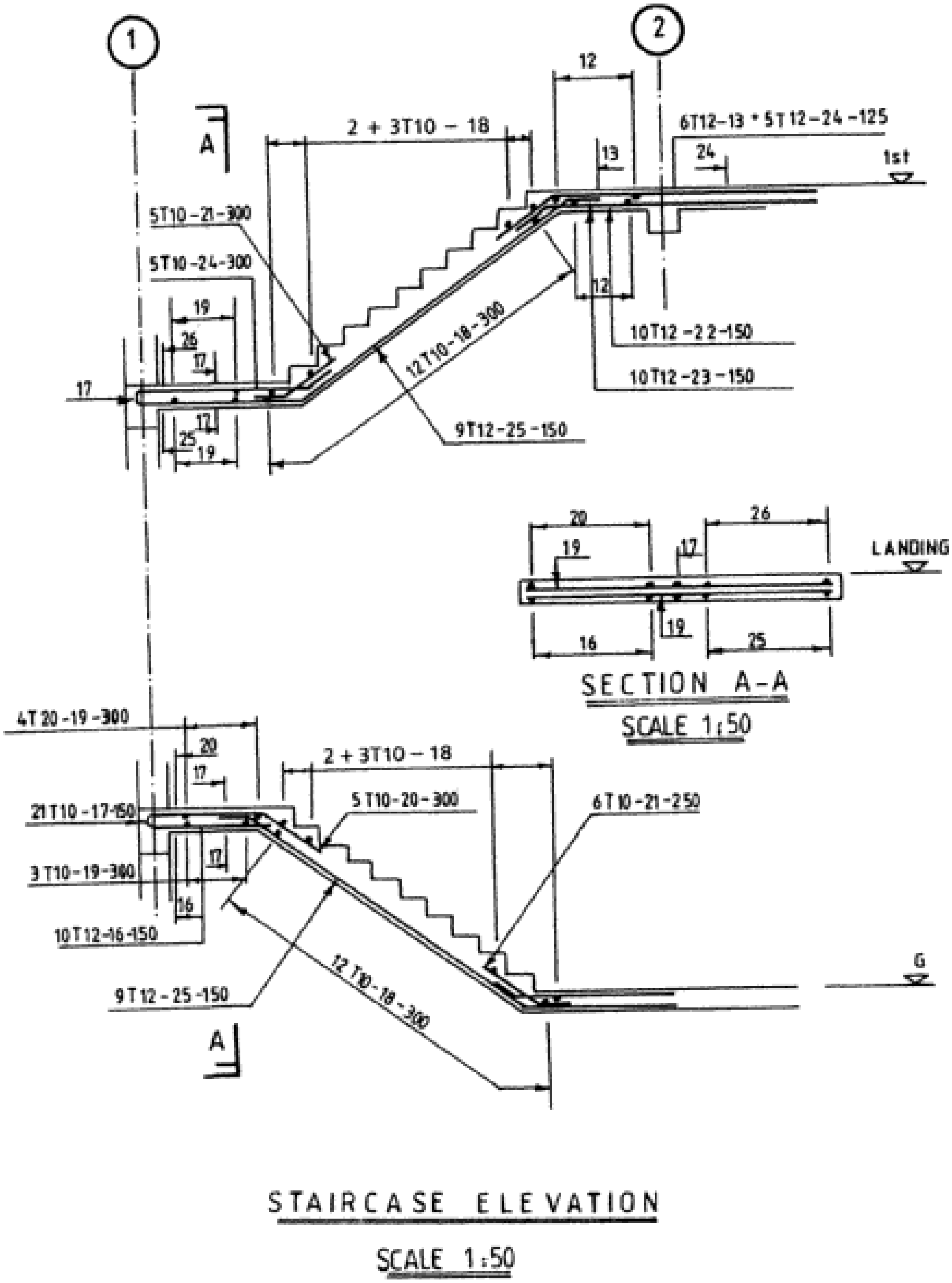


Figure A2.1.1. Staircase: Free-standing – Reinforcement details (British practice).

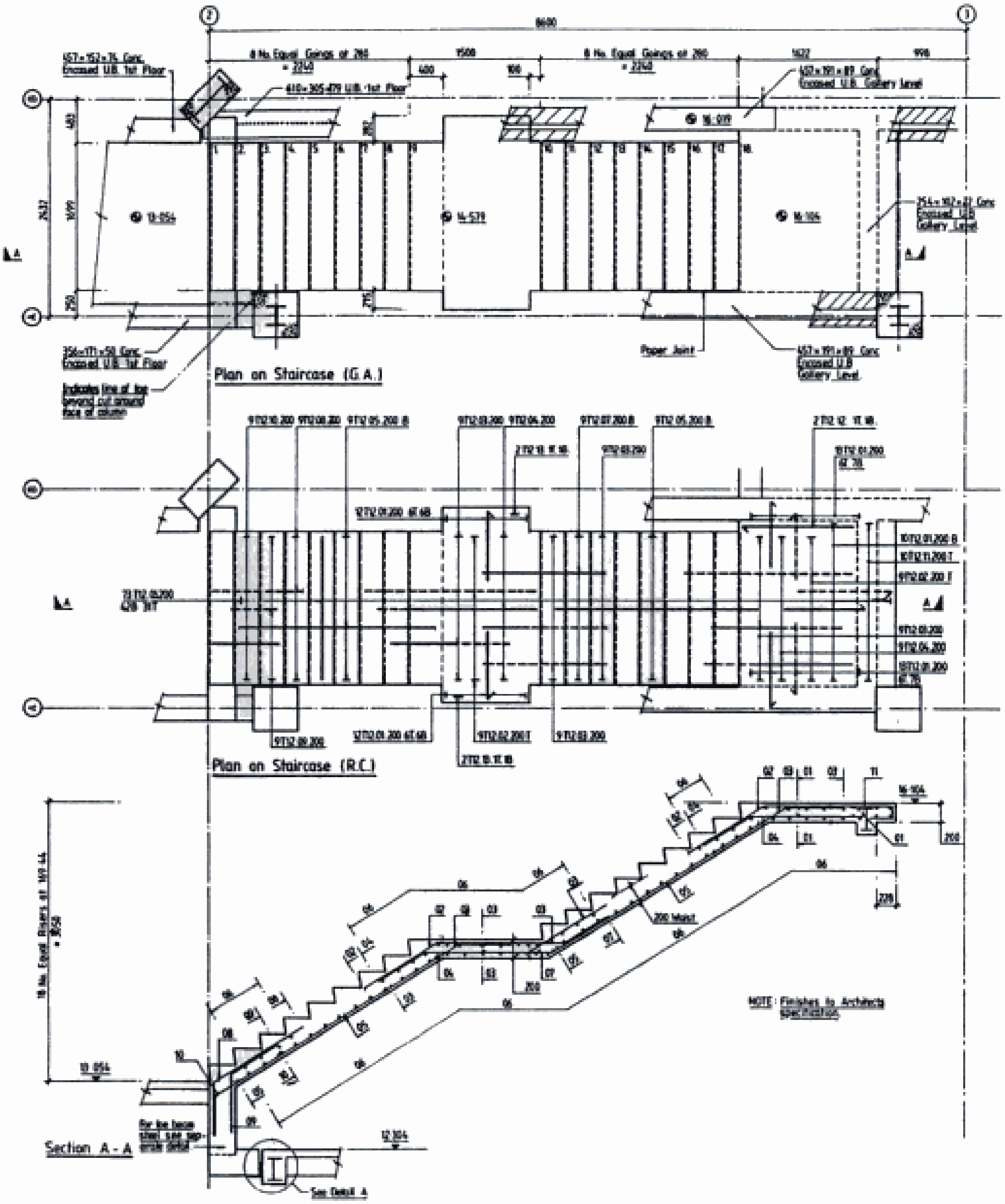


Figure A2.1.2. Staircase: Free-standing supported by brickwork or on beams – Reinforcement details (British practice).

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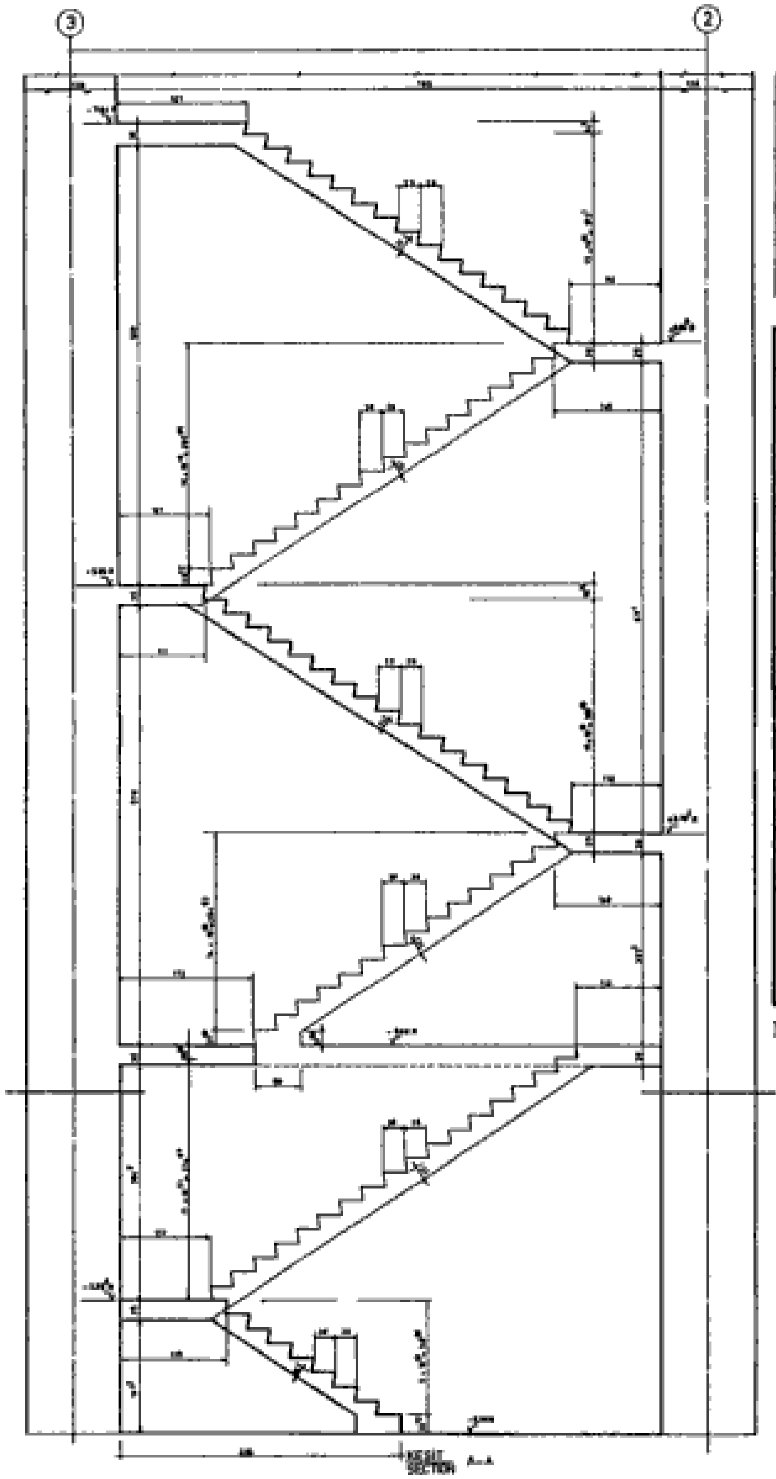

KAVİ

KABLO ve EMAYE BOBİN TELİ SANAYİ A.Ş. KARTAL TESİSLERİ

architects, engineers & consultants inc

TASARIM VE YAPIM GEREKLERİ			
BETON (Teknik şartname, bölüm E.2.1)			
BETON SINIFI	EN AZ DOZAJ	KÜP NUMUNENİN EN AZ KIRILMA MUKAVEMETİ	
		7 GÜNLÜK	28 GÜNLÜK
B80	150 kg/m^3	Gerekmez	80 kg/cm^2
B120	250 "	"	150 "
B160	300 "	140 kg/cm^2	200 "
B225	" "	195 "	275 "
B300	350 "	245 "	350 "
Kullanılacak Beton Sınıfı : B 300			

BETONARME ÇELİĞİ (Teknik şartname, bölüm E.2.3)				
ÇELİK KALİTESİ	ÇAP (mm)	EN AZ ÇEKME MUKAVEMETİ		EN AZ BİRİM KOPMA UZAMASI
		KOPMA	AKMA	
St IIIa/b	18'e kadar	4950 kg/cm^2	4200 kg/cm^2	8 %
"	18'den sonra	4800 "	4000 "	"

En az bindirme boyu : 50d (düz çubuklar için)
: 40d (kancalı çubuklar için)
En az pas payı : 2,5 cm.
Kaynaklı ve manşonlu bindirme yapmak yasaktır.

Note: GÜNLÜK in Turkish language is 'day strength'

REFERANS PAFTALARI / REFERENCE DRAWINGS	
PAFTA NO / NO OF DRWG.	PAFTANIN ADI / DESCRIPTION
820,97, ZD.4001	KORKULUK DET. – HANDRAIL DETAILS
821,95, Z A.4036	MERDİVEN DONATI DET. – STAIR REIN DET.

Figure A2.1.5. Plans and elevations of R.C. stairs: STEPS (Turkish/European practice).

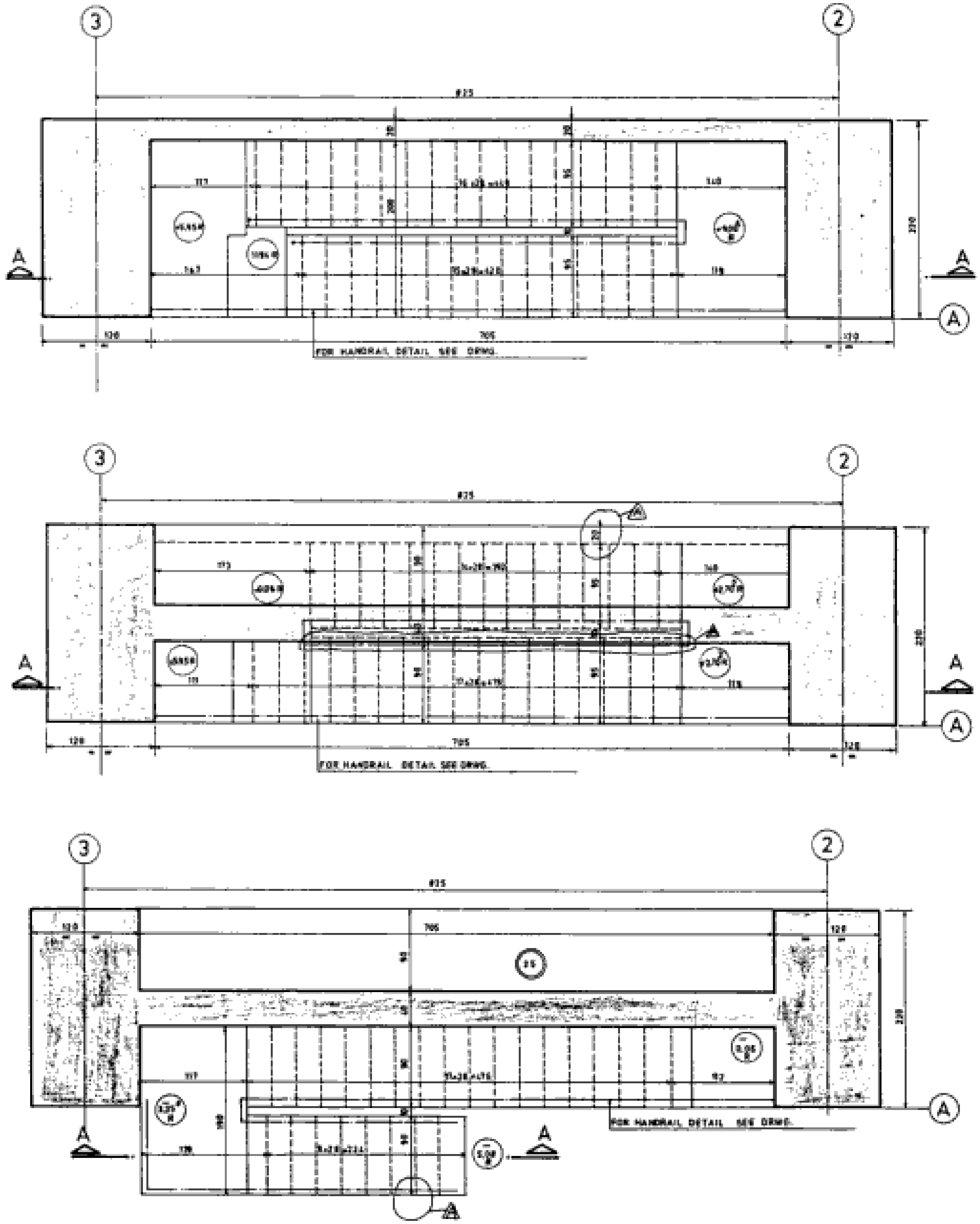

Figure A2.1.5 (cont.).

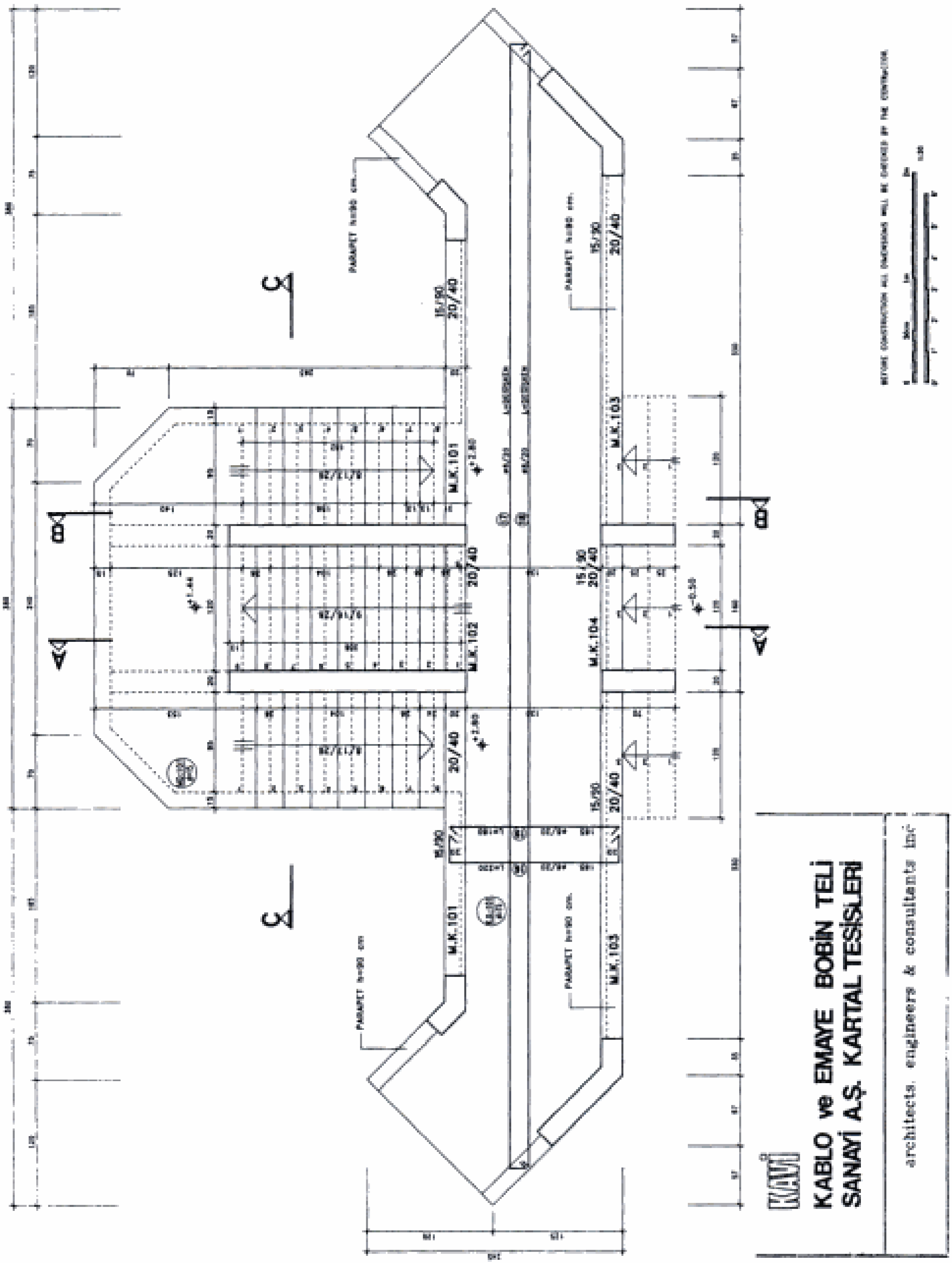

Figure A2.1.6. Typical reinforcement details of stairs and landings in a building: STEPS (Turkish/European practice).

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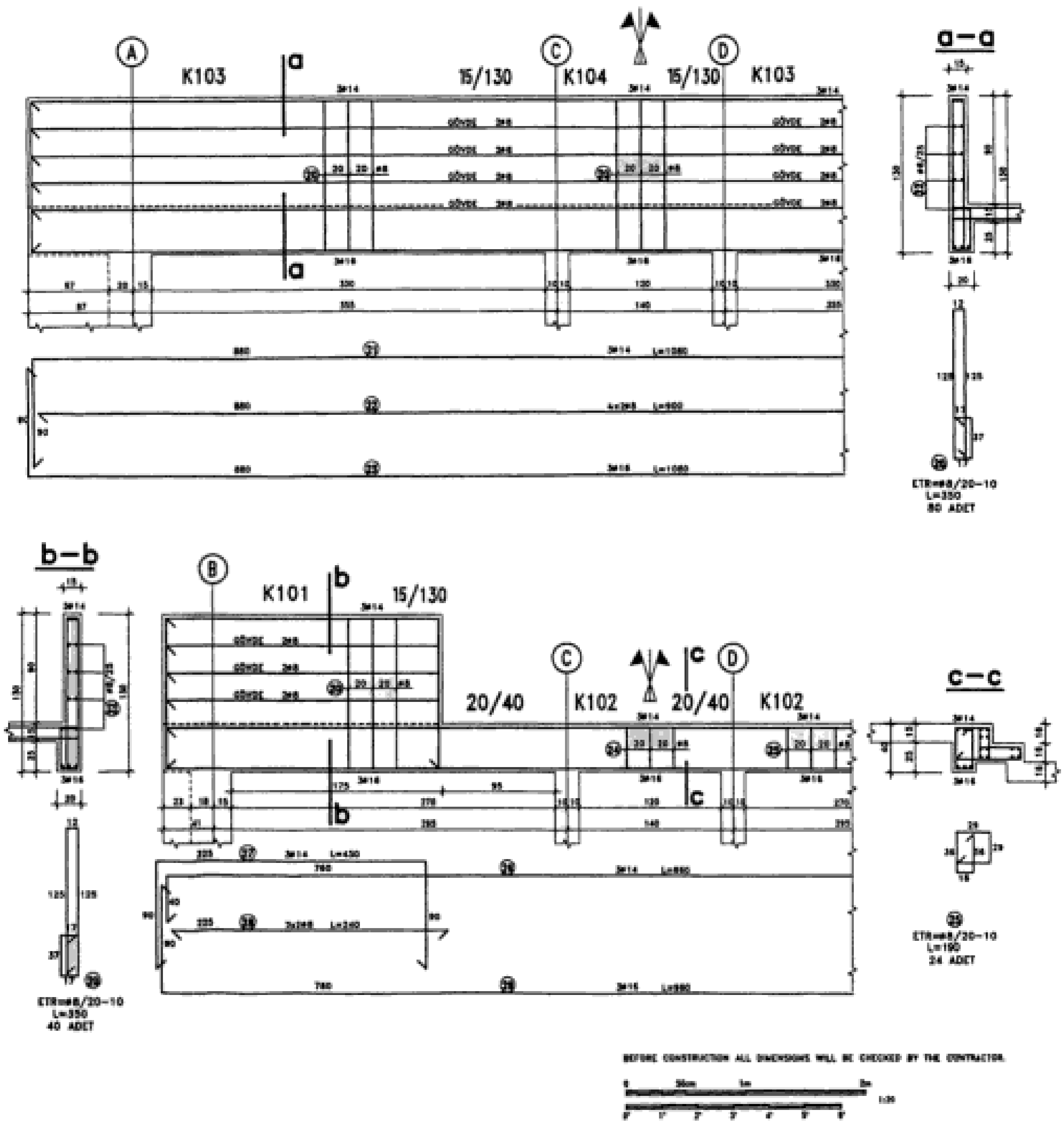

Figure A2.1.6 (cont.).

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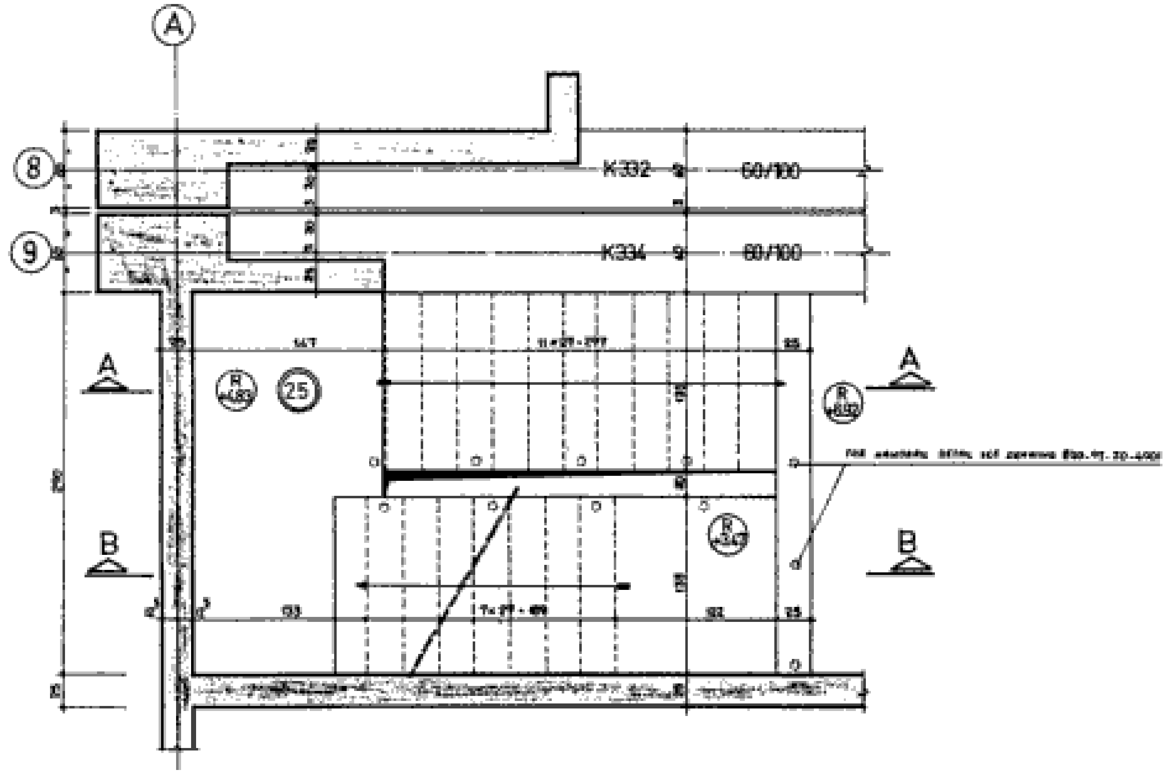
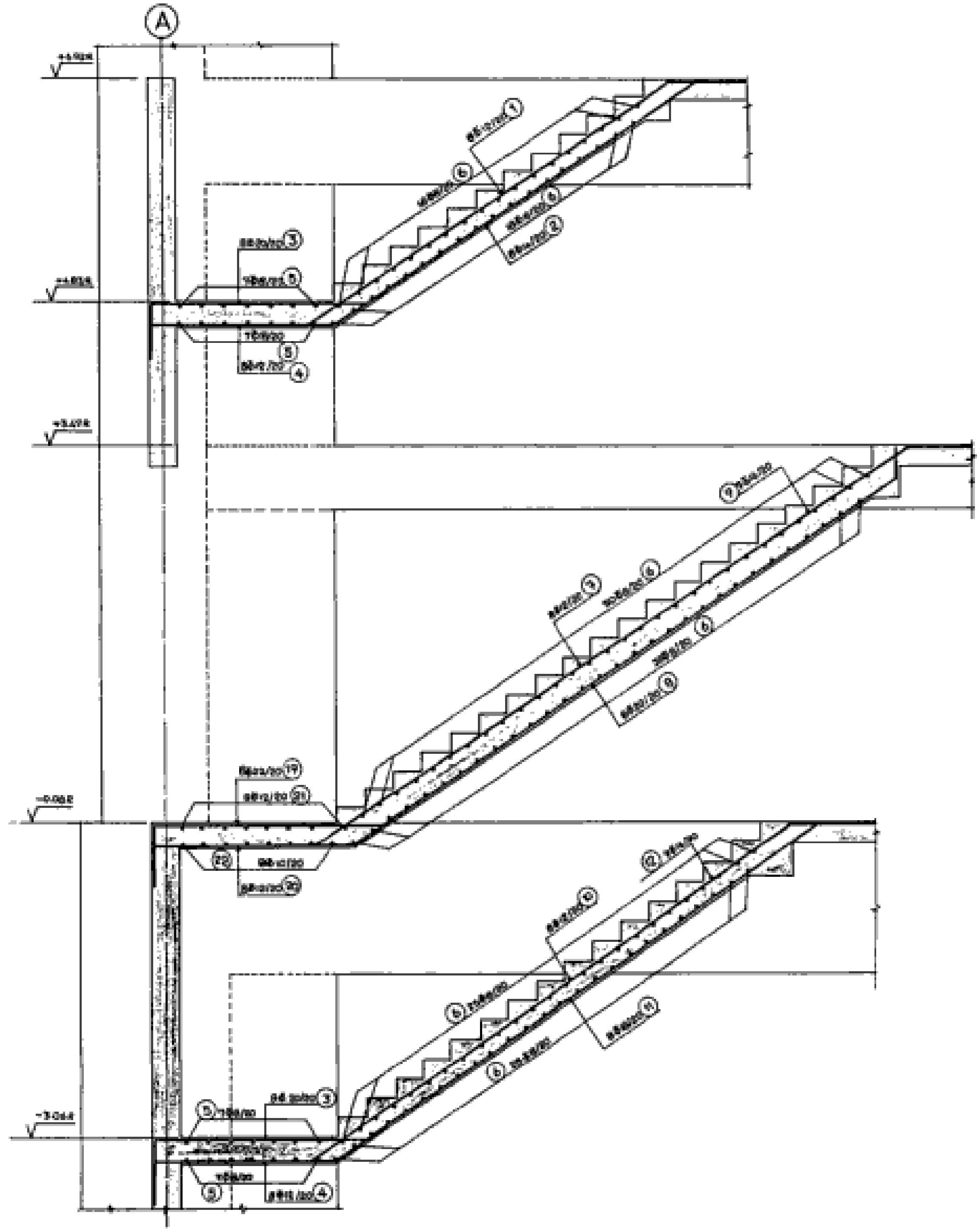


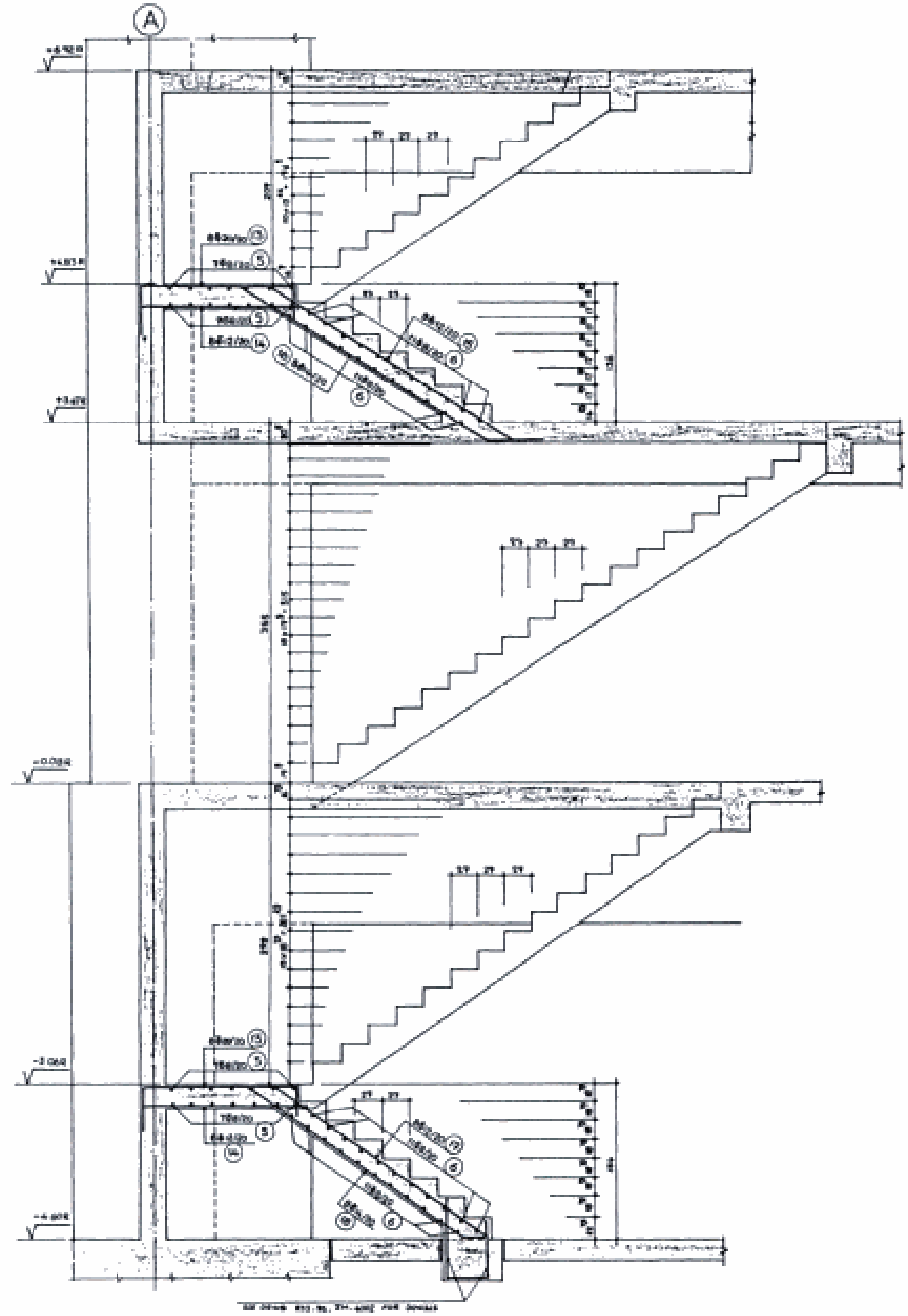
STAIRS BETWEEN LEVEL +3.47 & +6.92

Figure A2.1.7 (cont.).



SECTION A – A

Figure A2.1.7 (cont.).



SECTION B – B

Figure A2.1.7 (cont.).

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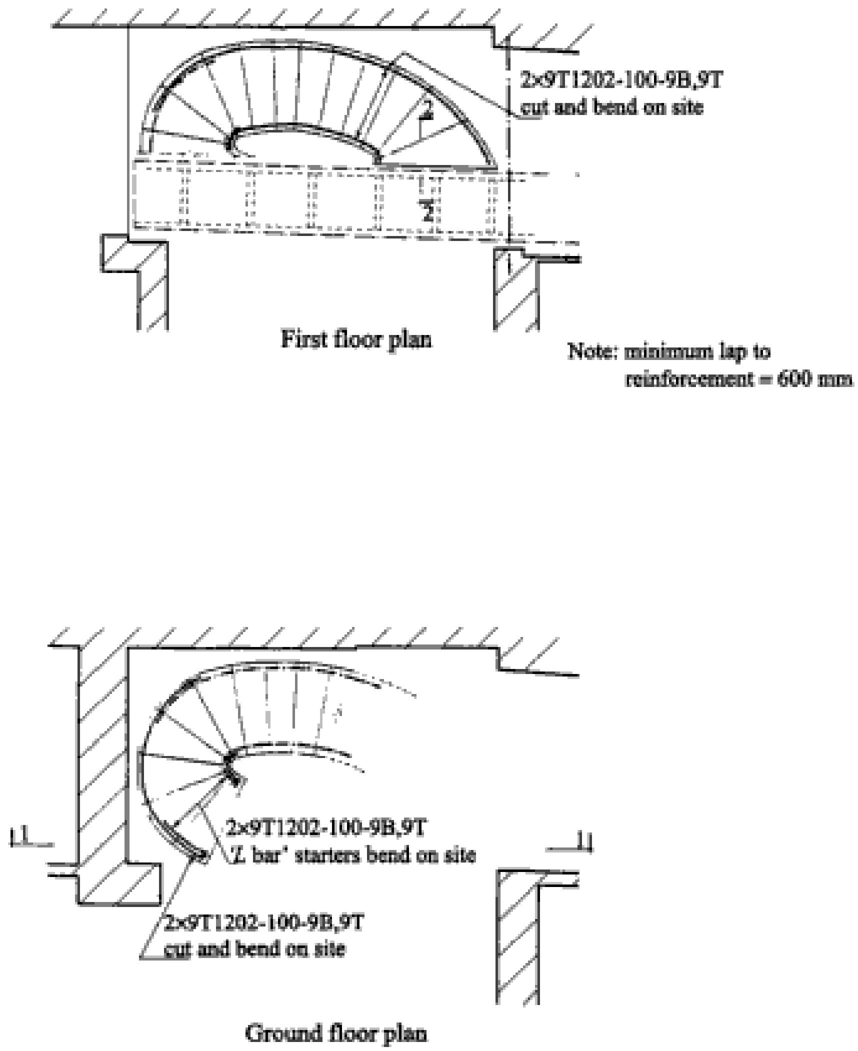

Figure A2.1.8.
Ellipto-helical staircase (Hyder Group UK) (British practice).

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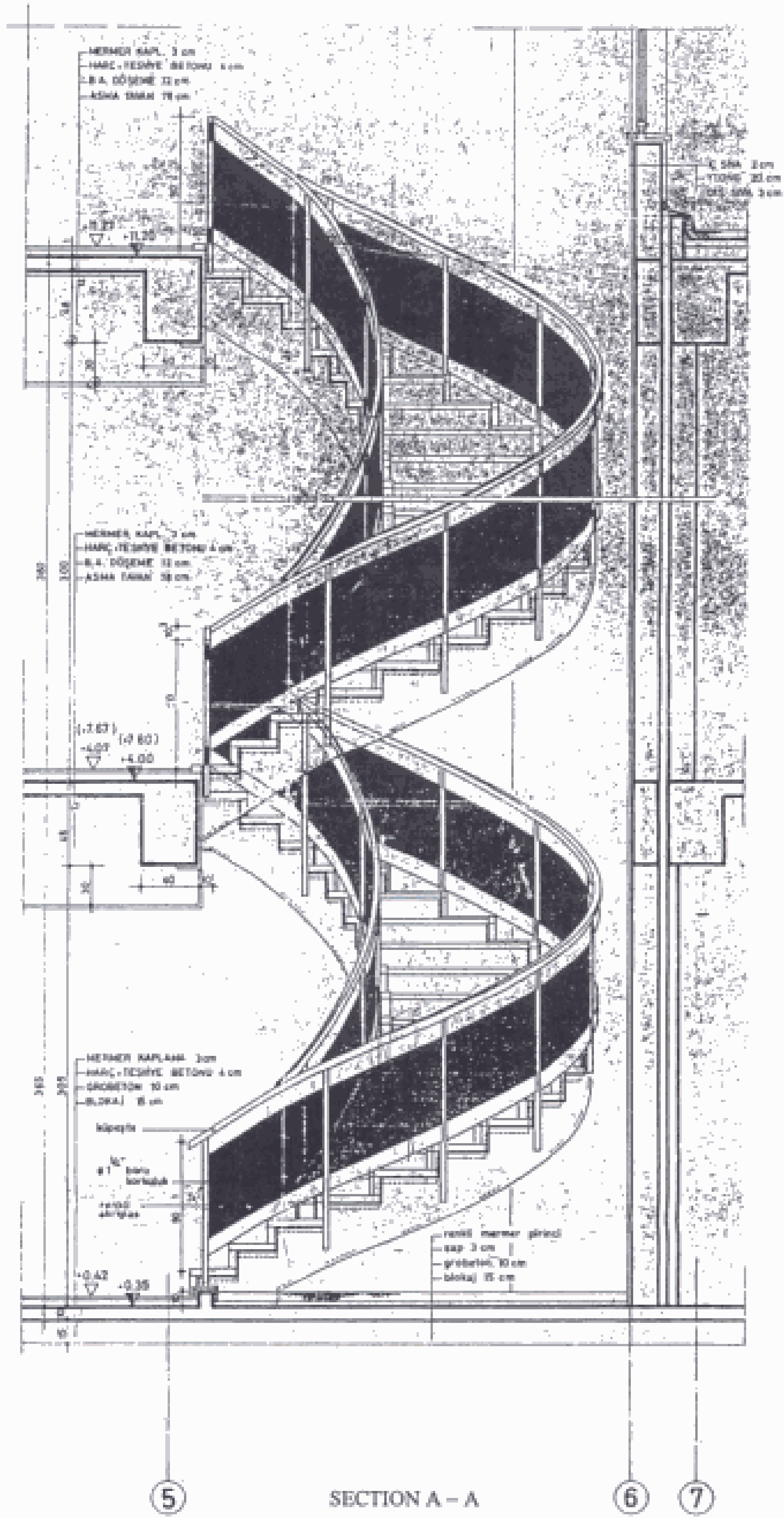

SECTION A – A

Figure A2.1.10. Helical staircase – Elevation and plans (Turkish/European practice).

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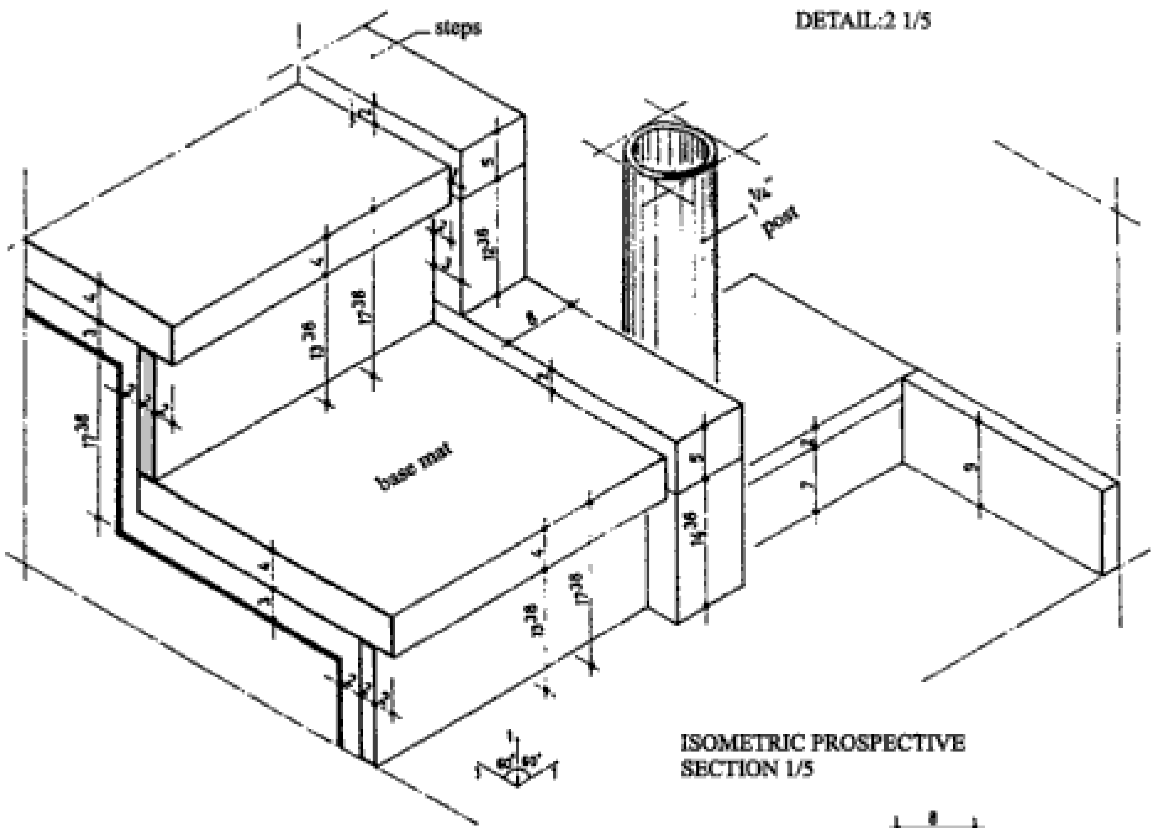
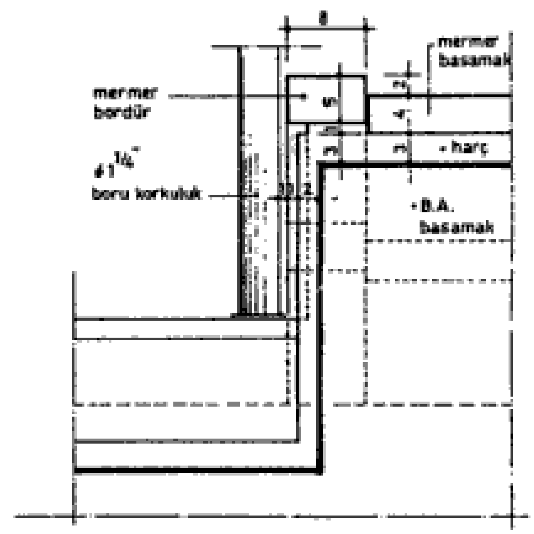
DETAIL:2 1/5

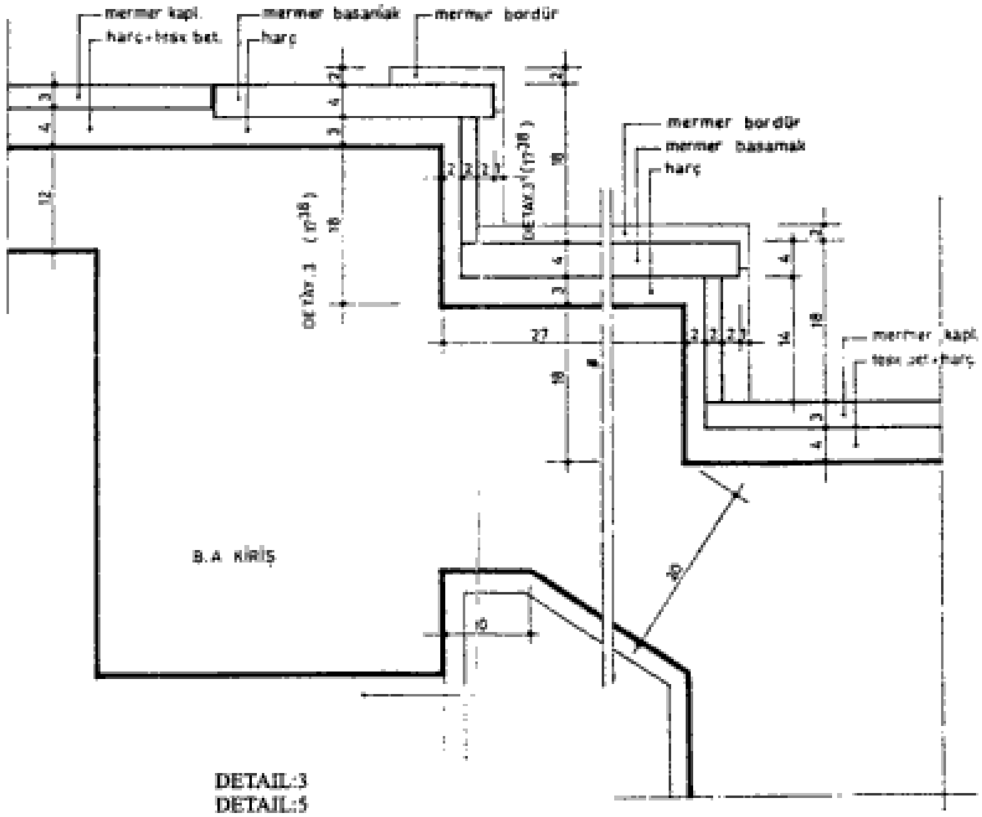


ISOMETRIC PROSPECTIVE
SECTION 1/5



II – II SECTION 1/5

Figure A2.1.11 (cont.).



DETAIL:3
DETAIL:5

DETAIL:4 1/5

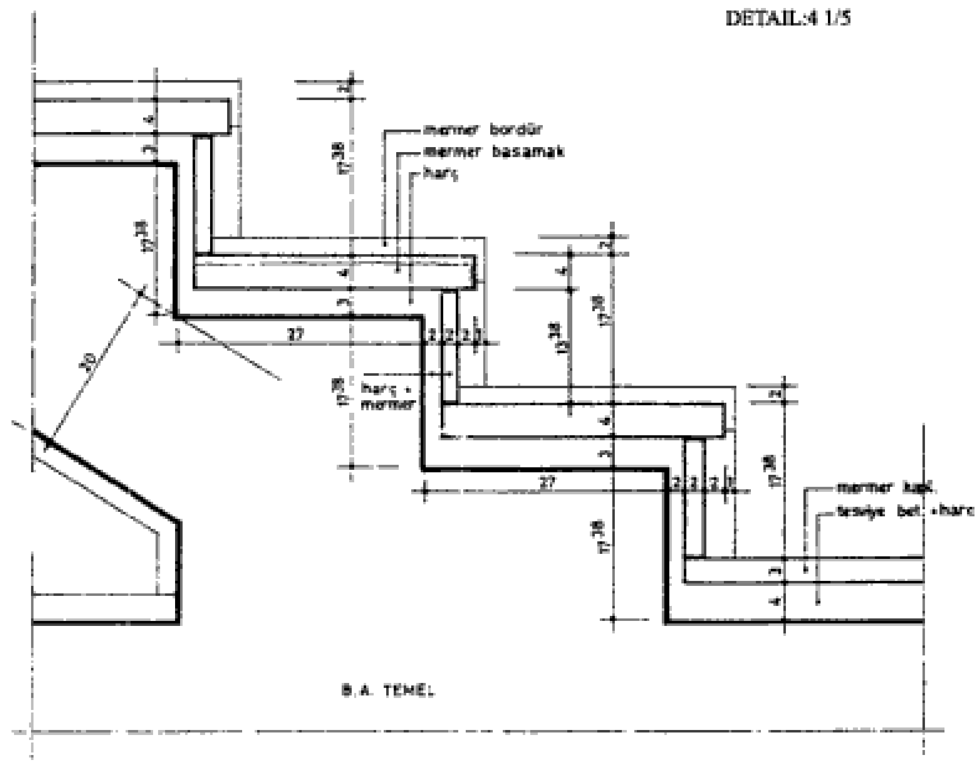
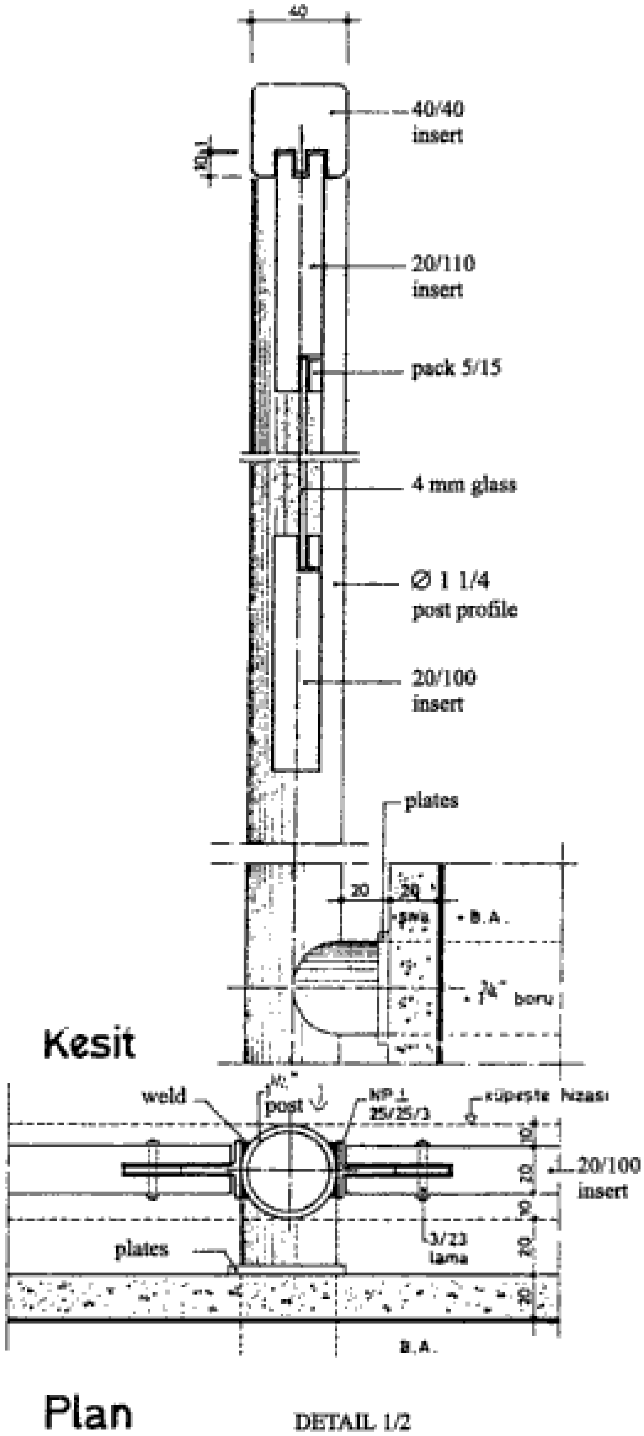
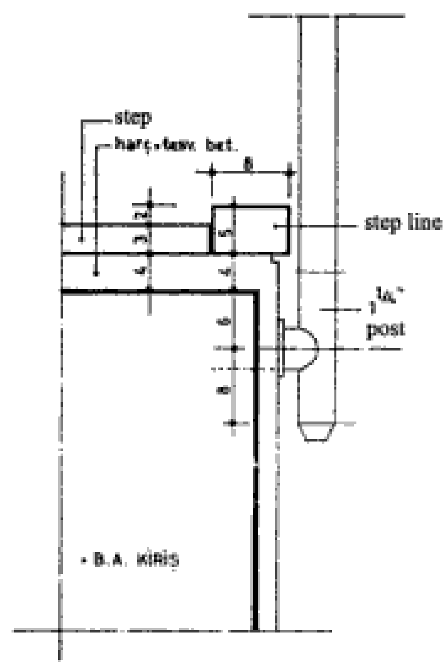


Figure A2.1.11 (cont.).

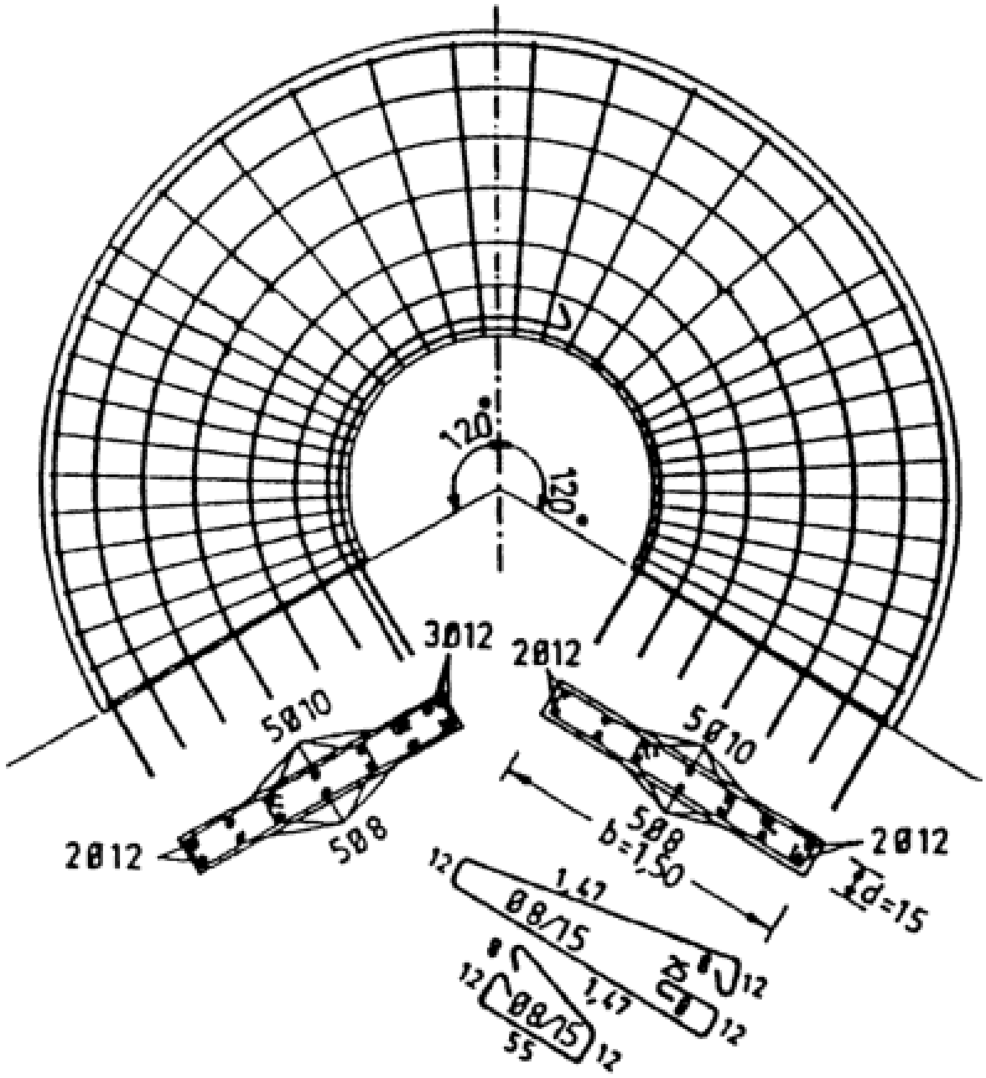


DETAIL 1/2



DETAIL:6 1/5

Figure A2.1.11 (cont.).



Note: 5∅10 is 5T10 British practice/American practice
∅8/15 is T8-150 mmc/c
b = 1,50 is b = 1.5 m etc.

Figure A2.1.12. Helical staircase – Circular in plan – Reinforcement details (European practice) (Von K. Winter 1977) (Ernst & Sohn).

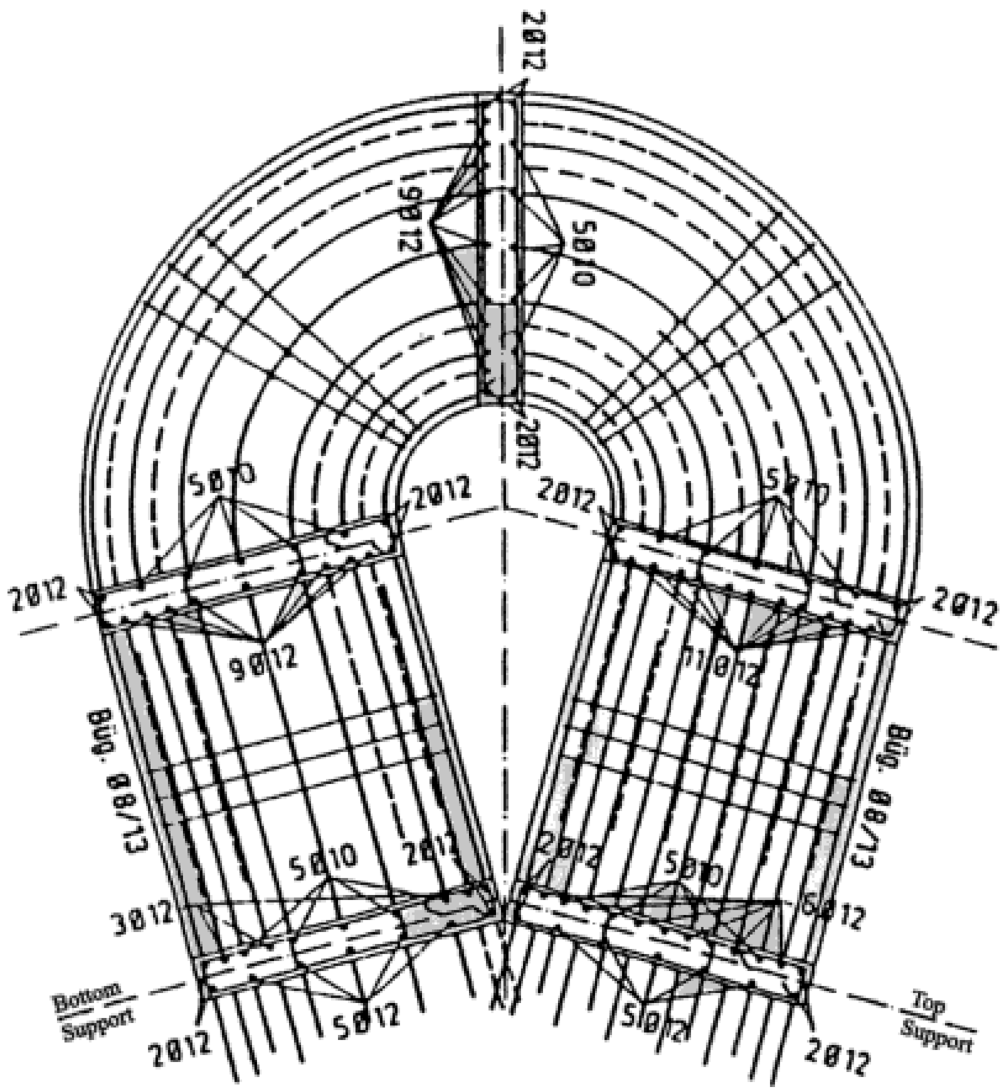


Figure A2.1.13. Helical-cum horseshoe staircase – Reinforcement details (see example) (German practice) Erlauterungen zu DIN 1080, Von K. Winter 1977, Ernst & Sohn (Compliments from Von K. Winter).

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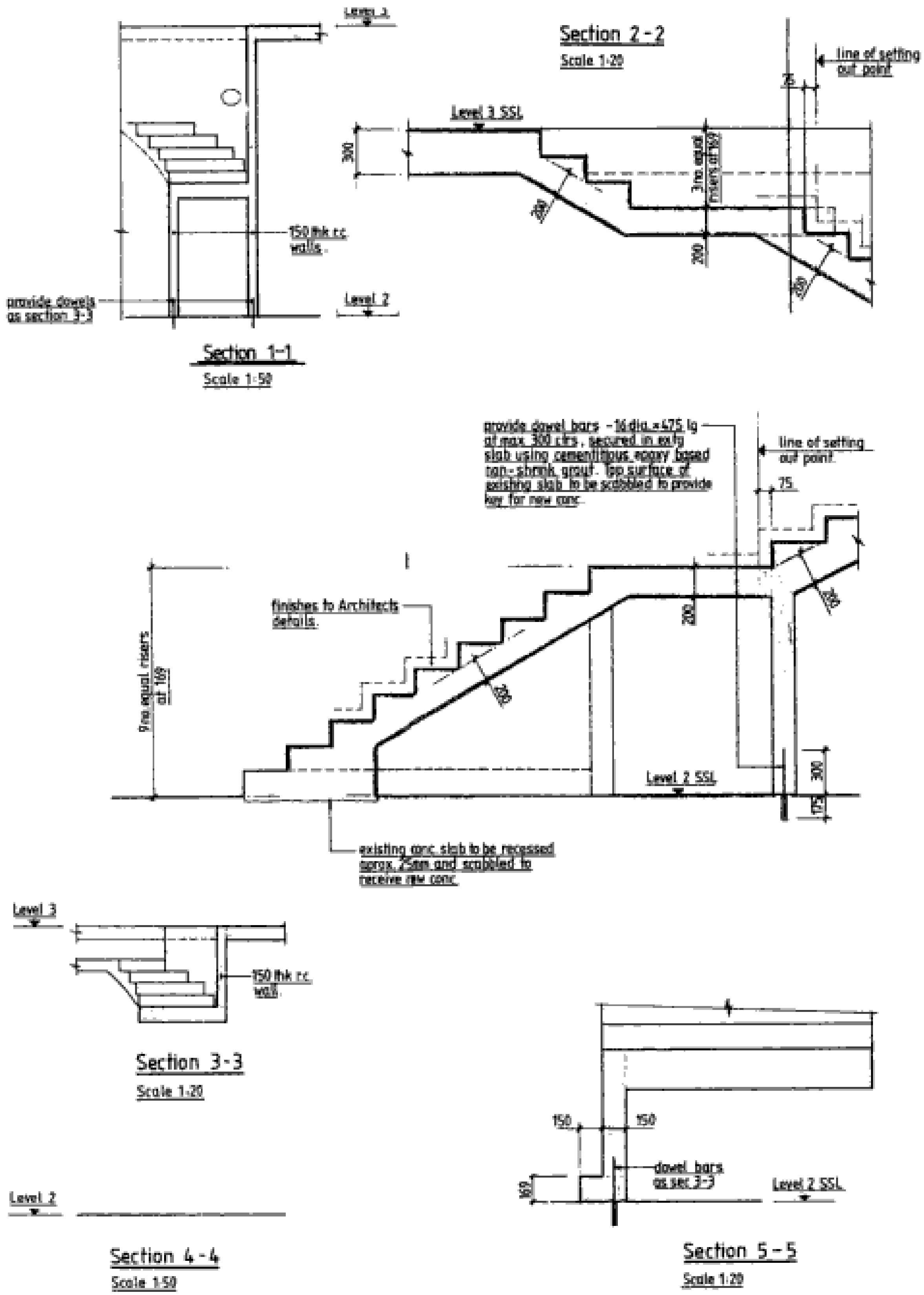

Figure A2.1.14 (cont.).

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NOTES

1. For general notes refer to DWG NO: 30/127/313.

2. Concrete to be Grade C35 with 20mm max. size of aggregate.
   All concrete to be in accordance with B.S.8110.

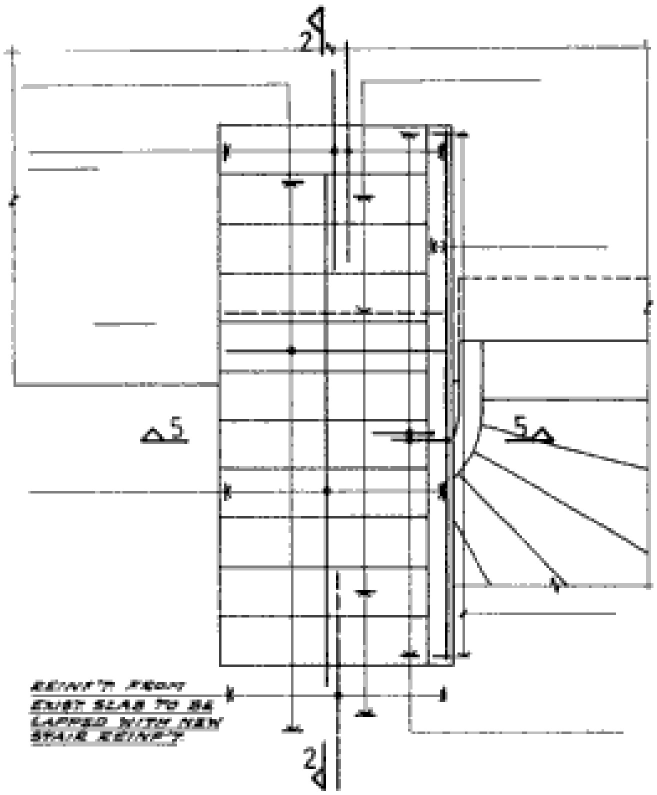
3. High yield steel, prefixed "T" on drawing and delivery tags, to be in accordance with B.S.4449 or B.S.4461.
   Mild steel, prefixed 'R' on drawing and delivery tags, to be in accordance with B.S.4449.
   High yield fabric to be in accordance with B.S.4483.

4. Minimum Laps
   T8 - 325
   T10 - 350
   T12 - 429

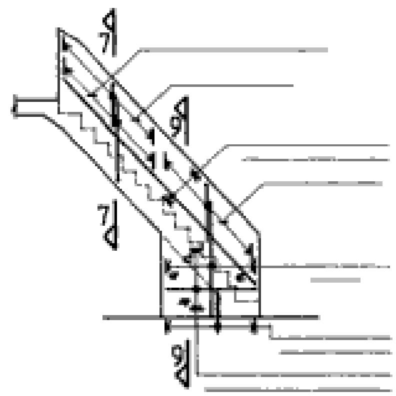
5. Cover to reinforcement to be 35mm U.N.O.

6. Handrail fixings to architects details.

7. For general arrangement refer to drawing no. 30/127/336.

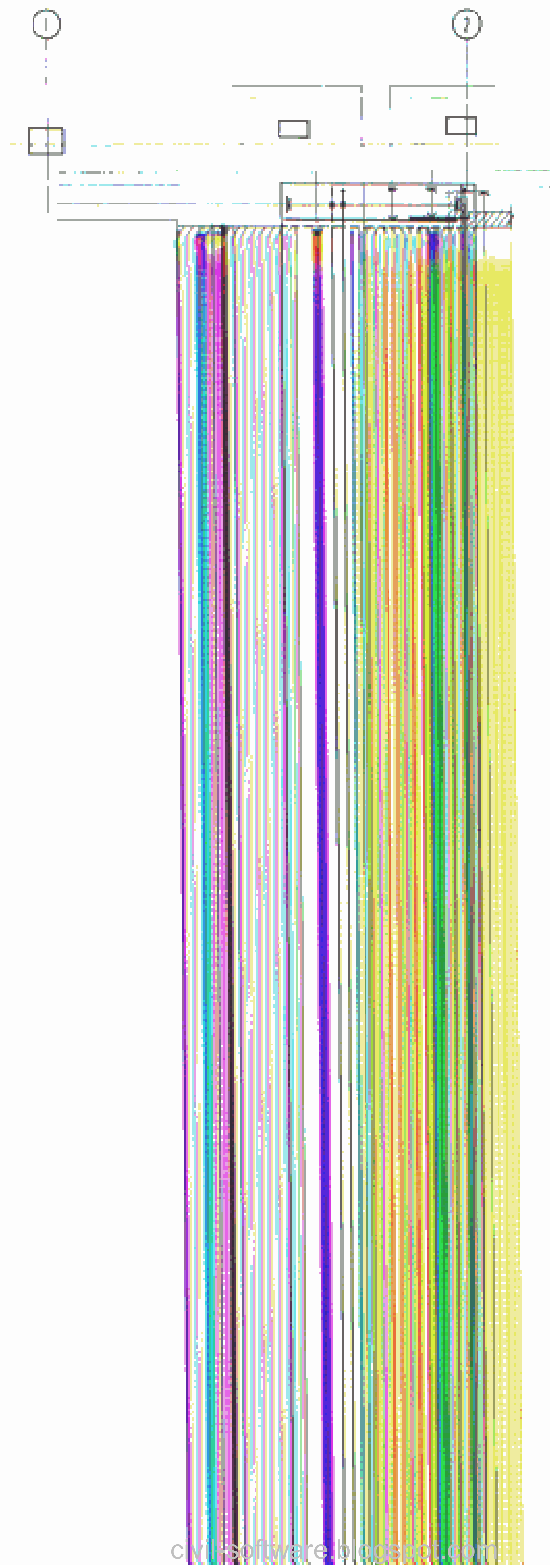


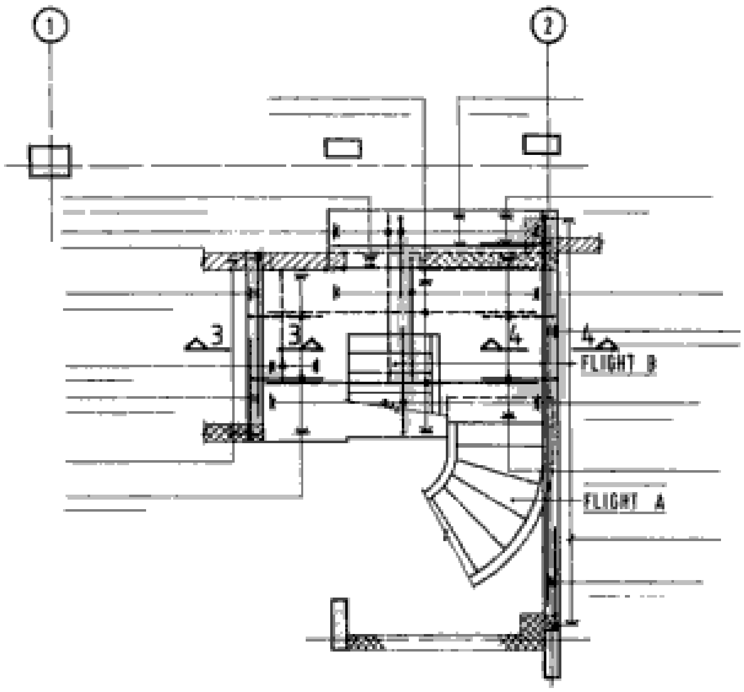
Plan on Flight B



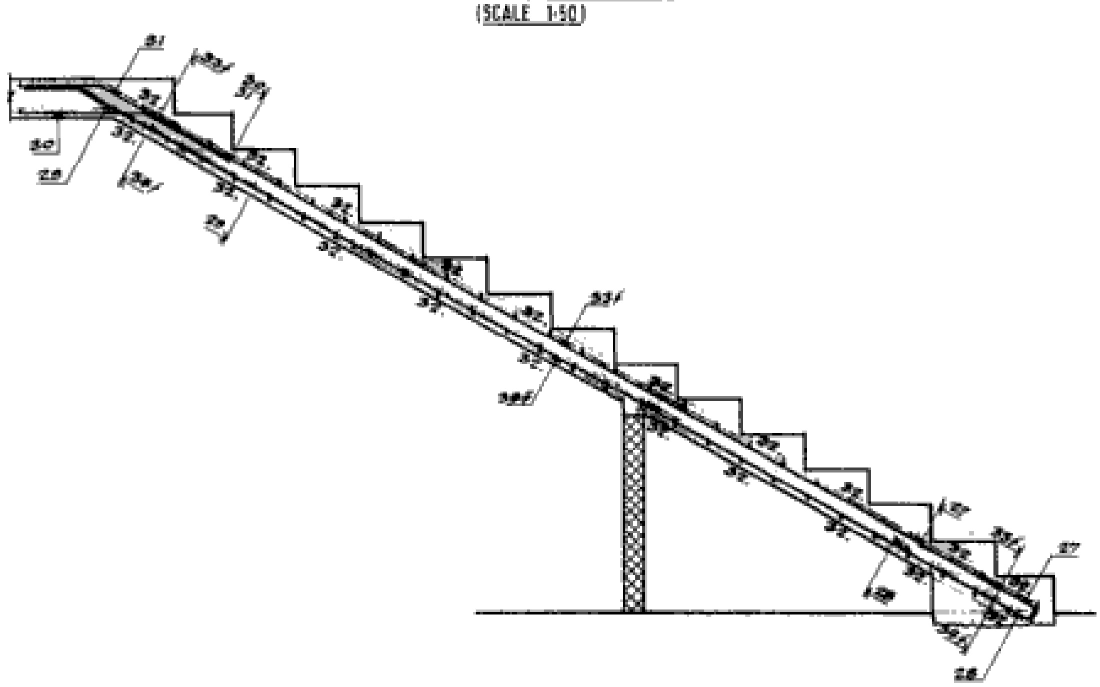
Elevation of Wall B
(Developed)
(SCALE 1:50)

Figure A2.1.15 (cont.).





Plan on Half Landing
(SCALE 1:50)



Section 1 — 1
(Developed)

Figure A2.1.15 (cont.).

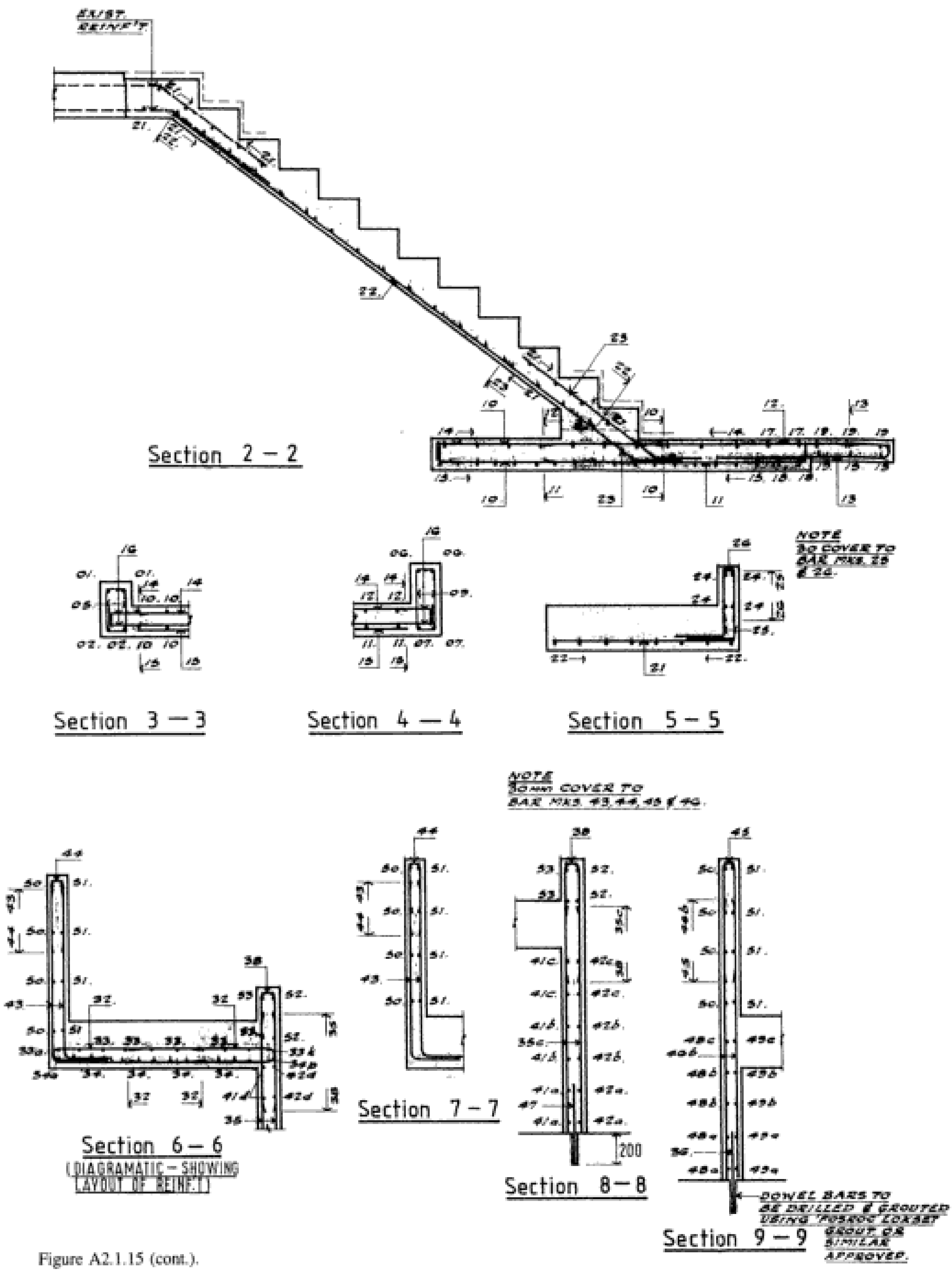
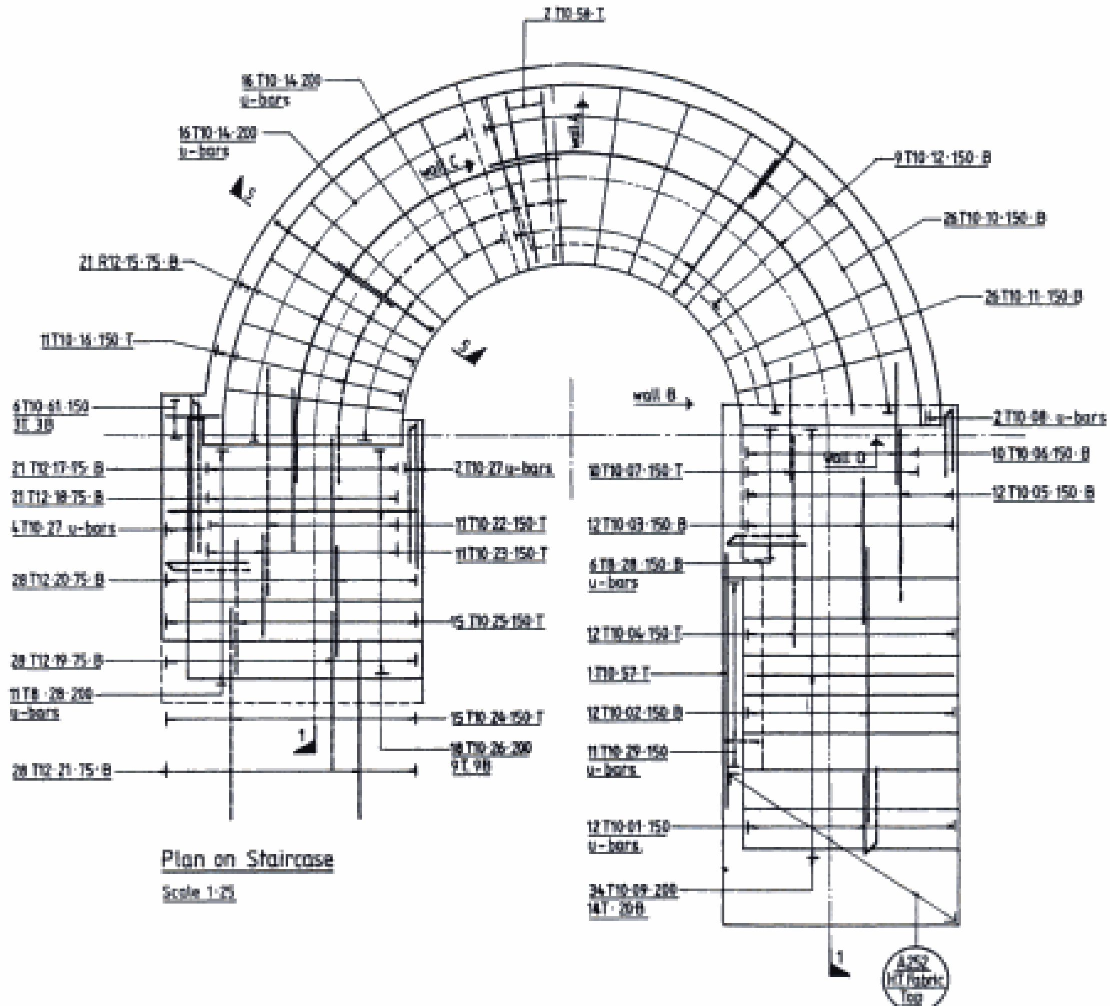


Figure A2.1.15 (cont.).



WARD :: COLE
consulting engineers
1 old lodge place st margarets twickenham
middlesex tw1 1rq
tel 0181 744 0900 fax 0181 744 0902

drawing title
R.C. Details of Staircase & Walls
Central Staircore – Level 2 to 3

scale	date	drawn by	checked by

NOTES:

1. For general notes refer to Ward + Cole drawing no 30/127/313.
2. For general arrangement refer to Ward + Cole drawing no: 30/12/355.
3. Concrete to be grade C35 with 20mm max. size of aggregate.
   All concrete to be in accordance with BS 8110
4. High yield steel, prefixed "T" on drawing and delivery tags, to be in accordance with BS 4449 OR BS 4461.
   Mild steel, prefixed "R" on drawing and delivery tags, to be in accordance with BS 4449
   High yield fabric to be in accordance with BS 4483
5. Minimum laps to reinforcement:
   T8 - 325
   T10 - 350
   T12 - 425
6. Concrete cover to reinforcement to be:
   Staircase - 35mm U.N.O.
   Walls - 25mm U.N.O.
7. The Engineer is to inspect the fixed reinforcement prior to pouring concrete. Contractor to give Ward + Cole min. 48 hours prior notice.

Figure A2.1.16. Mixed staircase – Straight-cum circular/helical (Ward & Cole, London) (British practice).

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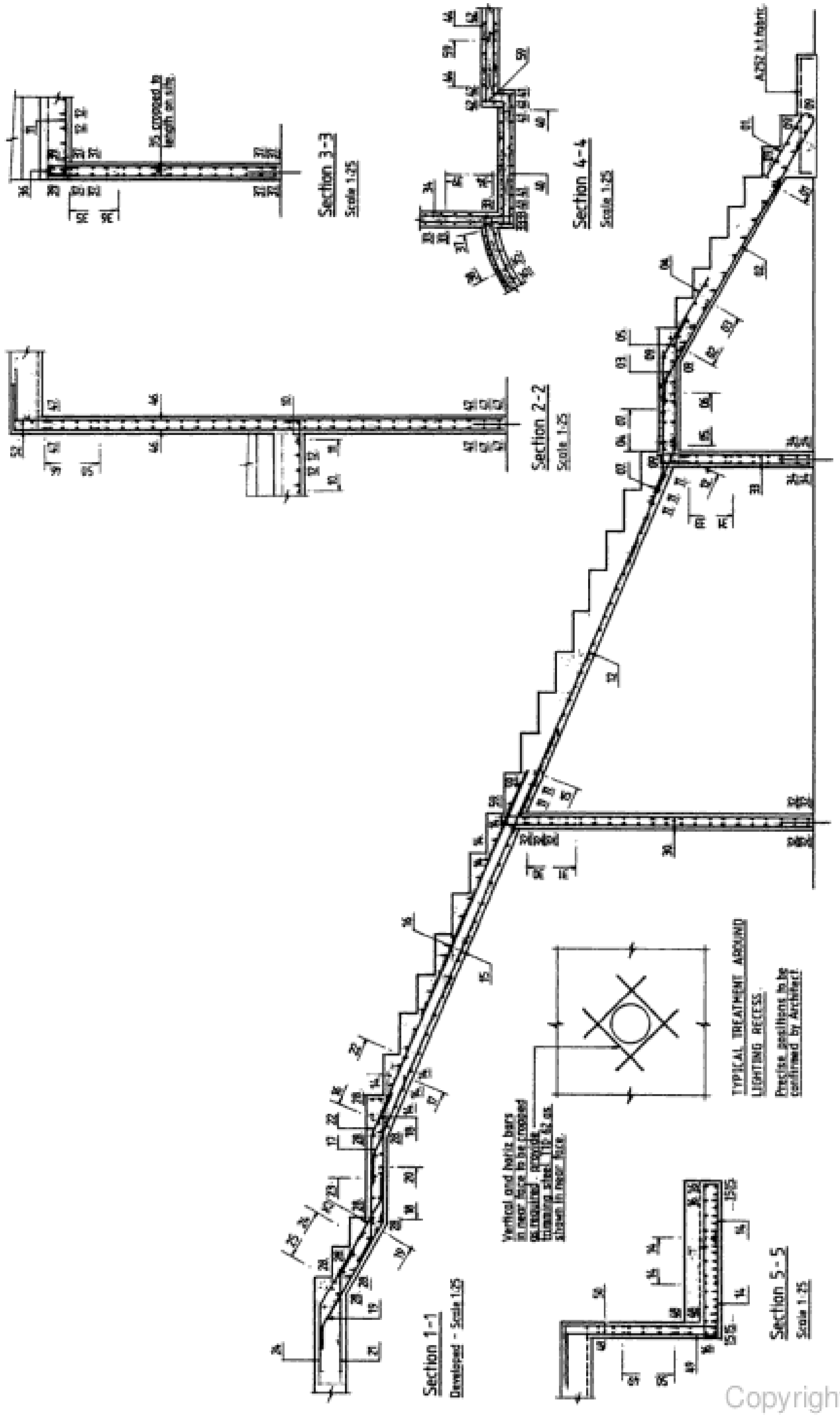

Figure A2.1.16 (cont.).

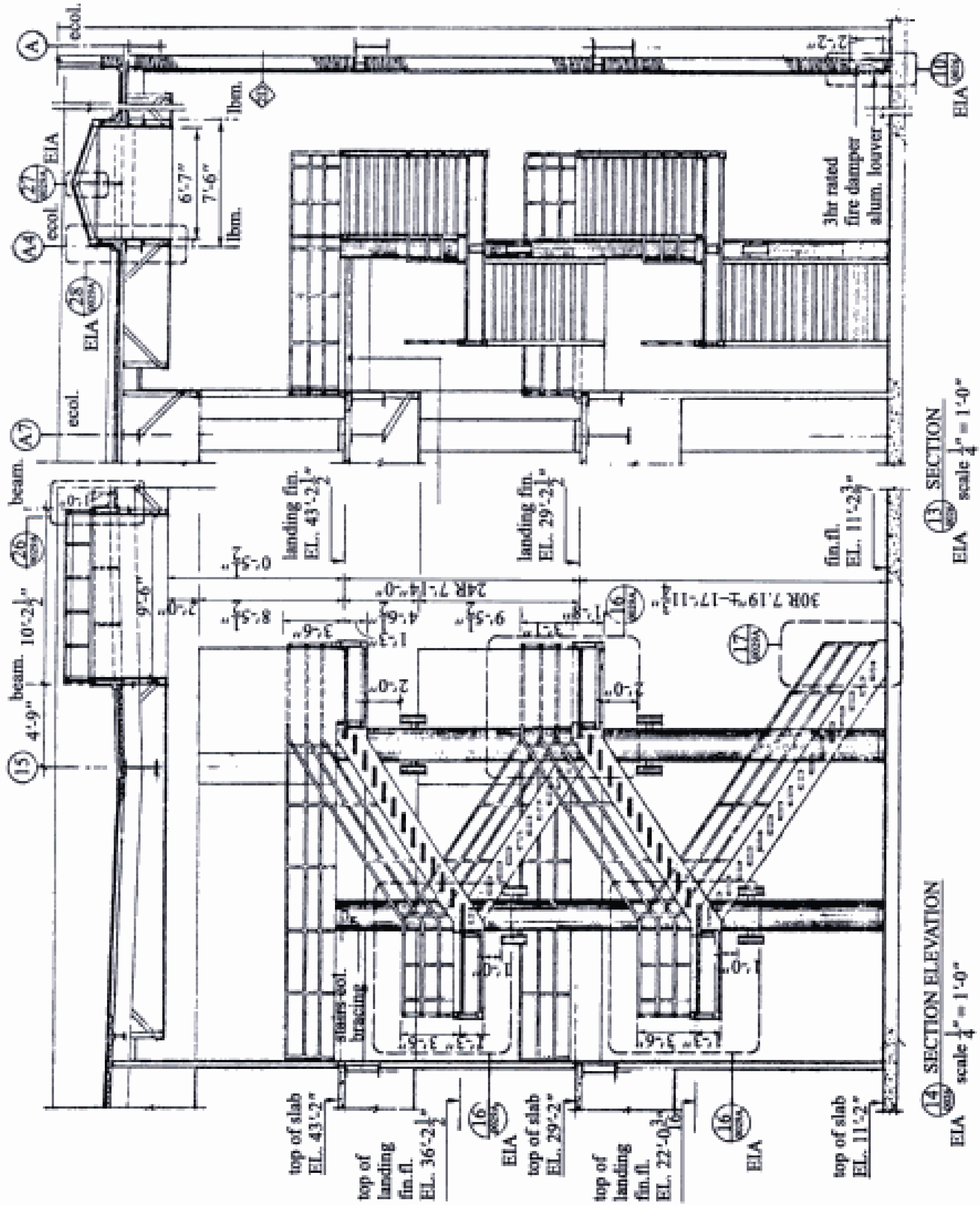

Figure A2.2.1. Sectional elevation of a steel stairs (Gibbs & Hill, New York) (American practice).

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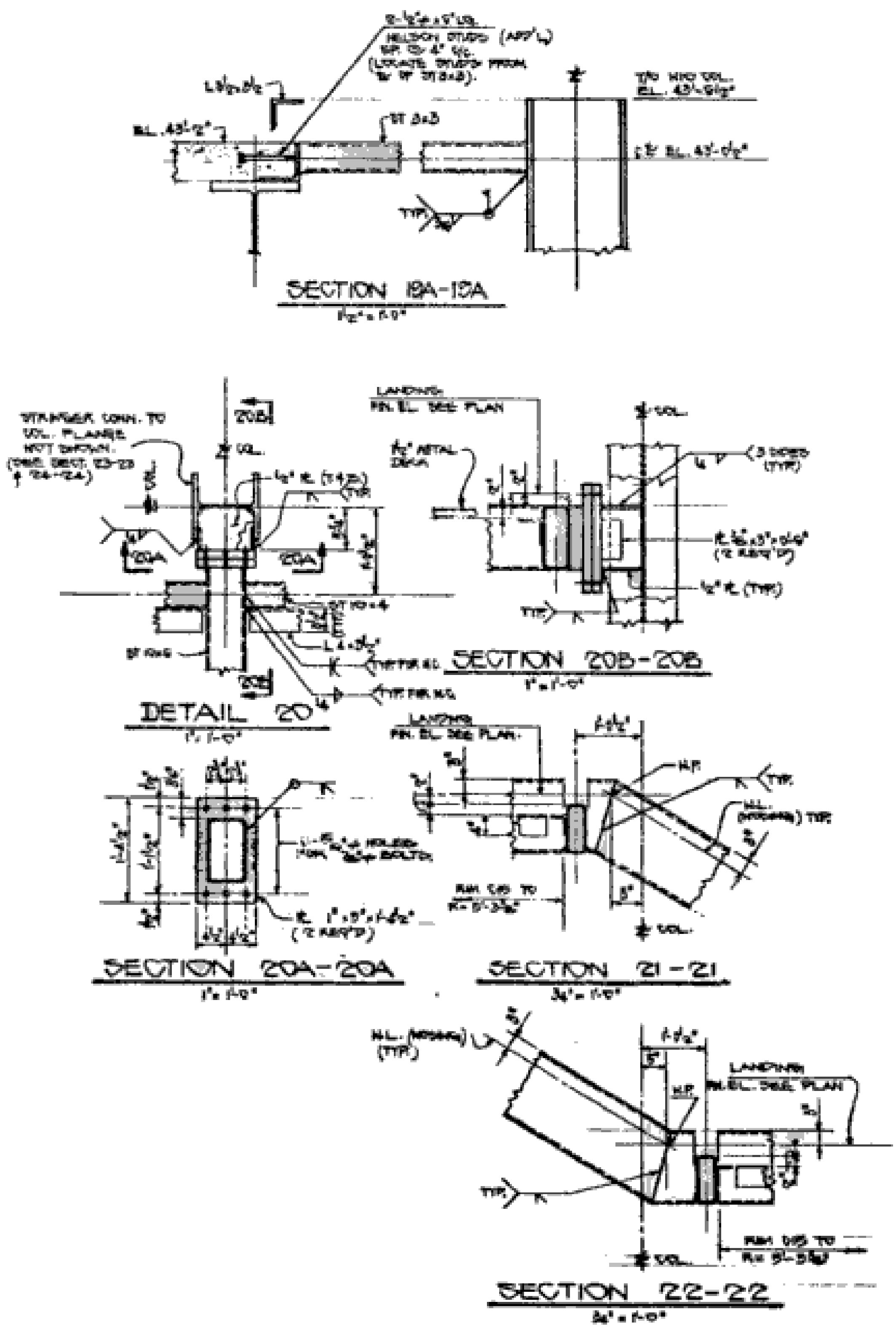
GROUND FLOOR
EL. 11'-2"
WORKING LINE
TOP OF FIN. FLR.
CHIP OFF CONC. SURFACE FOR STRINGER INSTALLATION, IF REQ'D.
HOLES
STRINGERS



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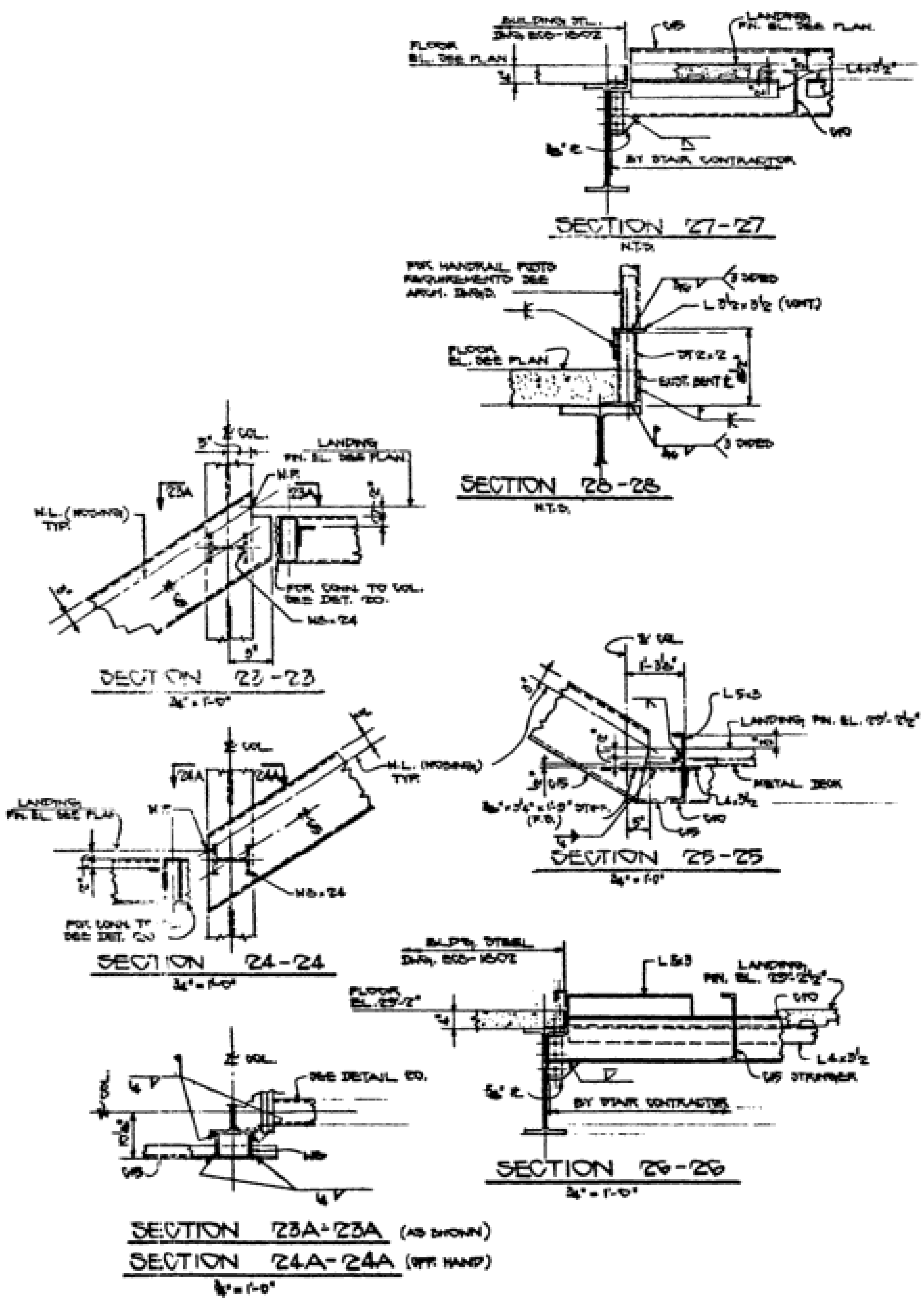

Figure A2.2.2 (cont.).

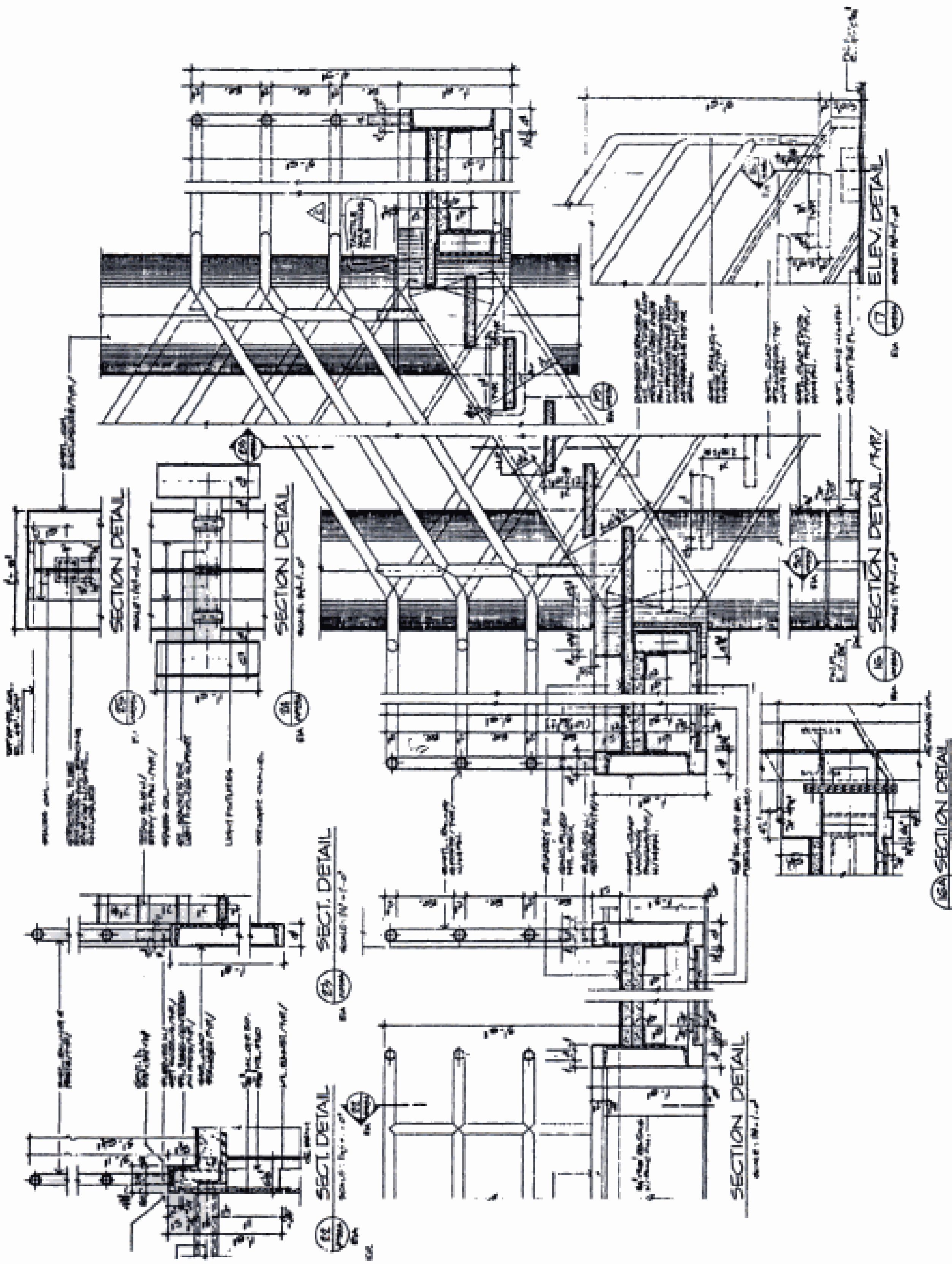

Figure A2.2.3. Arch details of stair (Gibbs & Hill, New York) (American practice).

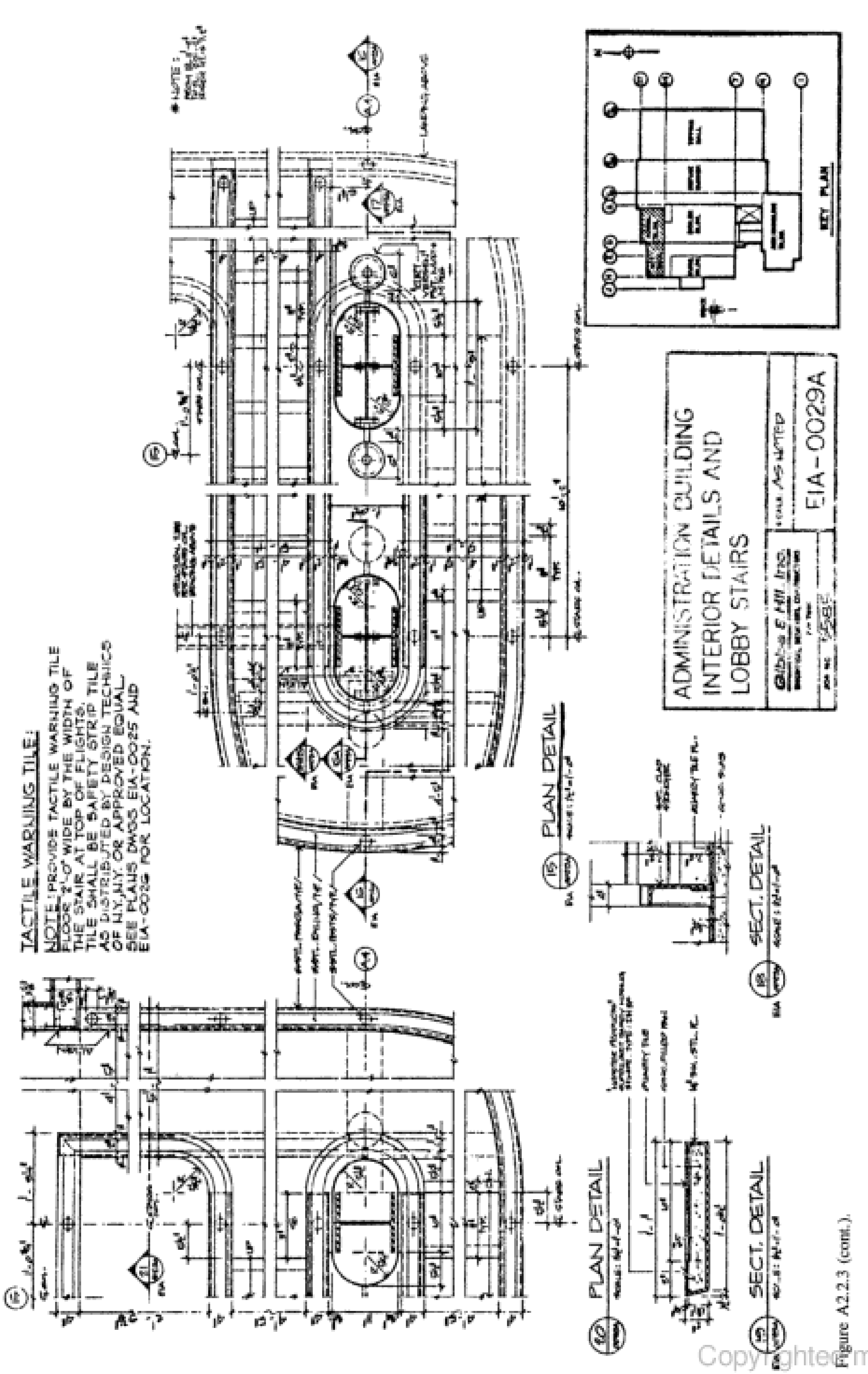

Figure A2.2.3 (cont.).

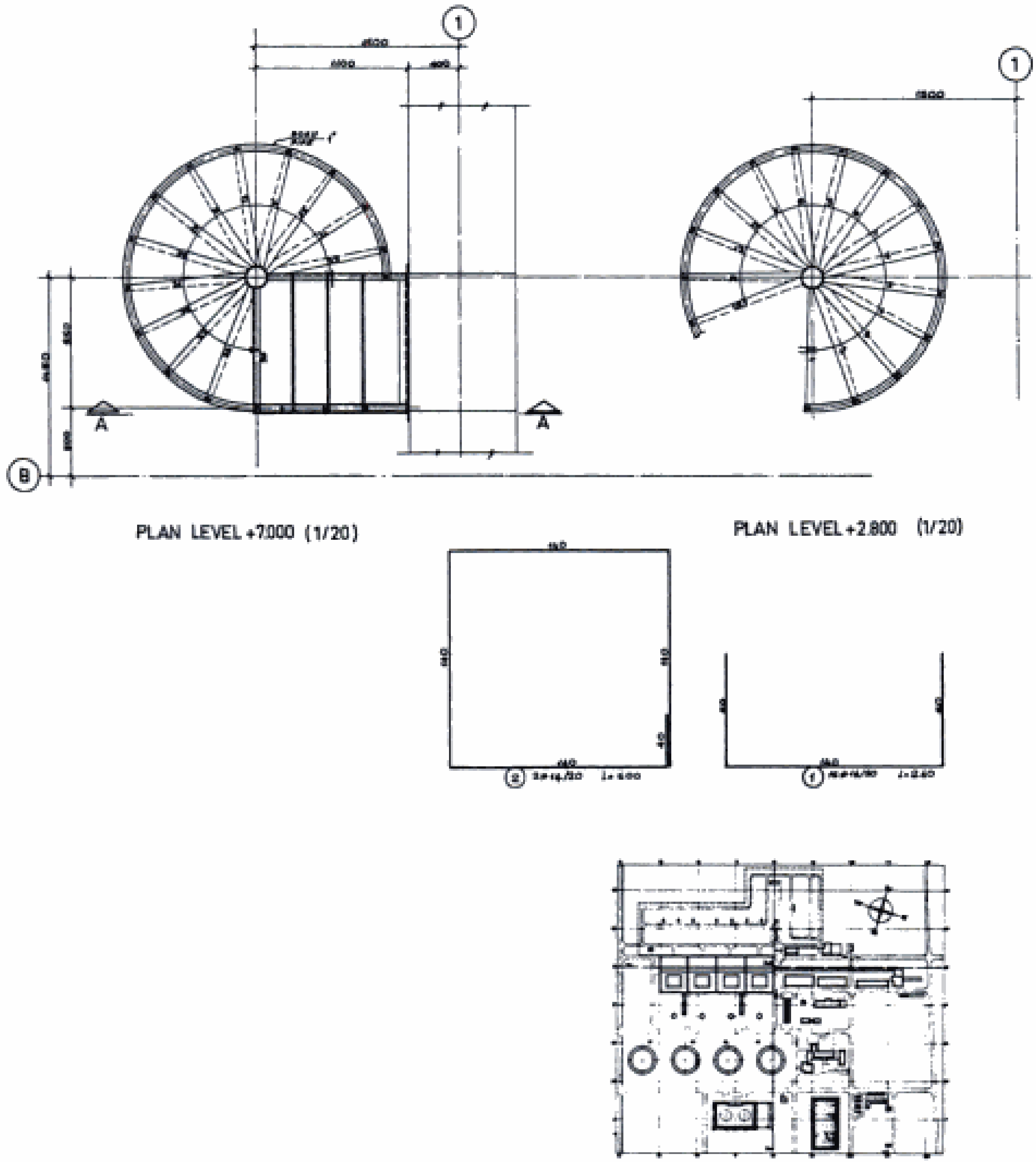

Figure A2.2.4. Steel helical staircase – Elevations, plans and structural details (Turkish/European practice).

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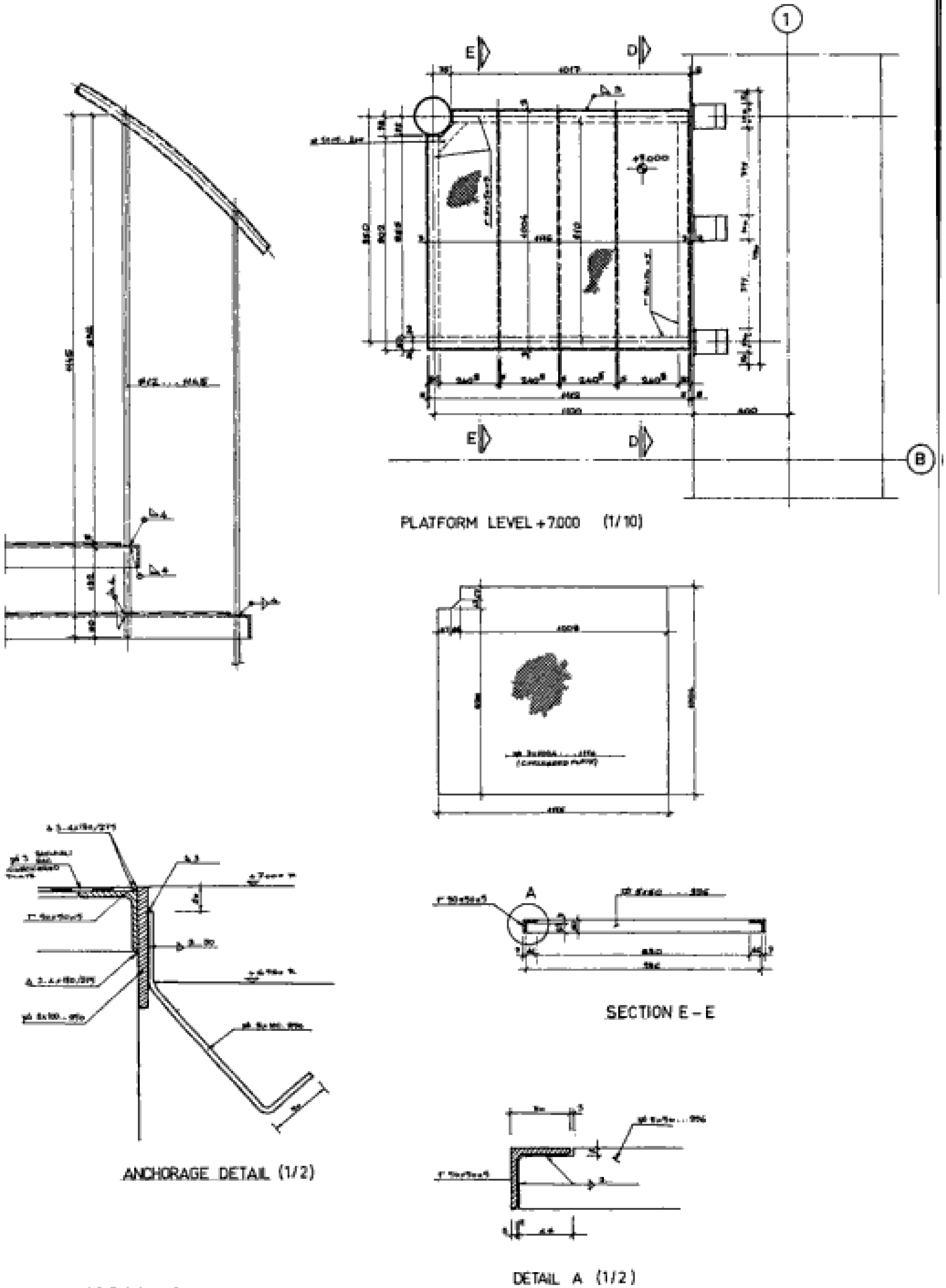

Figure A2.2.4 (cont.).

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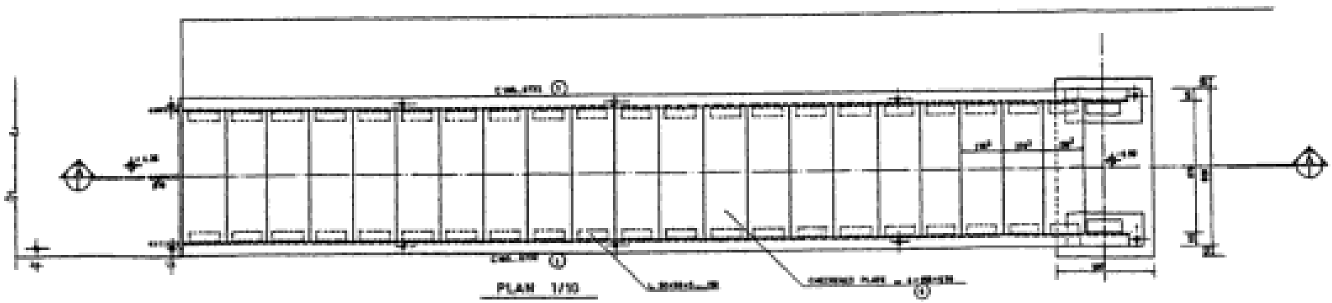

MARK	PROFILE TYPE	LENGTH	NUMBER	TOTAL LENGTH	UNIT WEIGTH	TOTAL WEIGHT
POZ	PROFİL CİNSİ	BOY	ADET	TOPLAM BOY	BİRİM AĞIRLIK	TOPLAM AĞIRLIK
1	C 160	6,732	2	13464	18.8	253.123
2	L 50X50X5	0.150	42	6.30	3.77	23.751
3	L 80X80X8	0.150	2	0.300	9.66	2.898
4	⌀ 10X230X800	0.0018	1		7.85	14.444
5	⌀ 10X165X400	0.0006	2	0.0006	7.85	10.362
6	⌀ 8X220X607	0.0008	21	0.0168	7.85	132.0[illegible]
TOTAL WEIGTH / TOPLAM AĞIRLIK Kg	≅ 436.662 kg (1 ADEDİ)					

ALL DIMENSIONS SHOWN ON THE DRAWINGS SHALL BE VERIFIED BY THE CONTRACTOR IN THE FIELD.

MATERIAL St 37

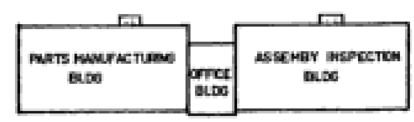

Figure A2.2.5 (cont.).

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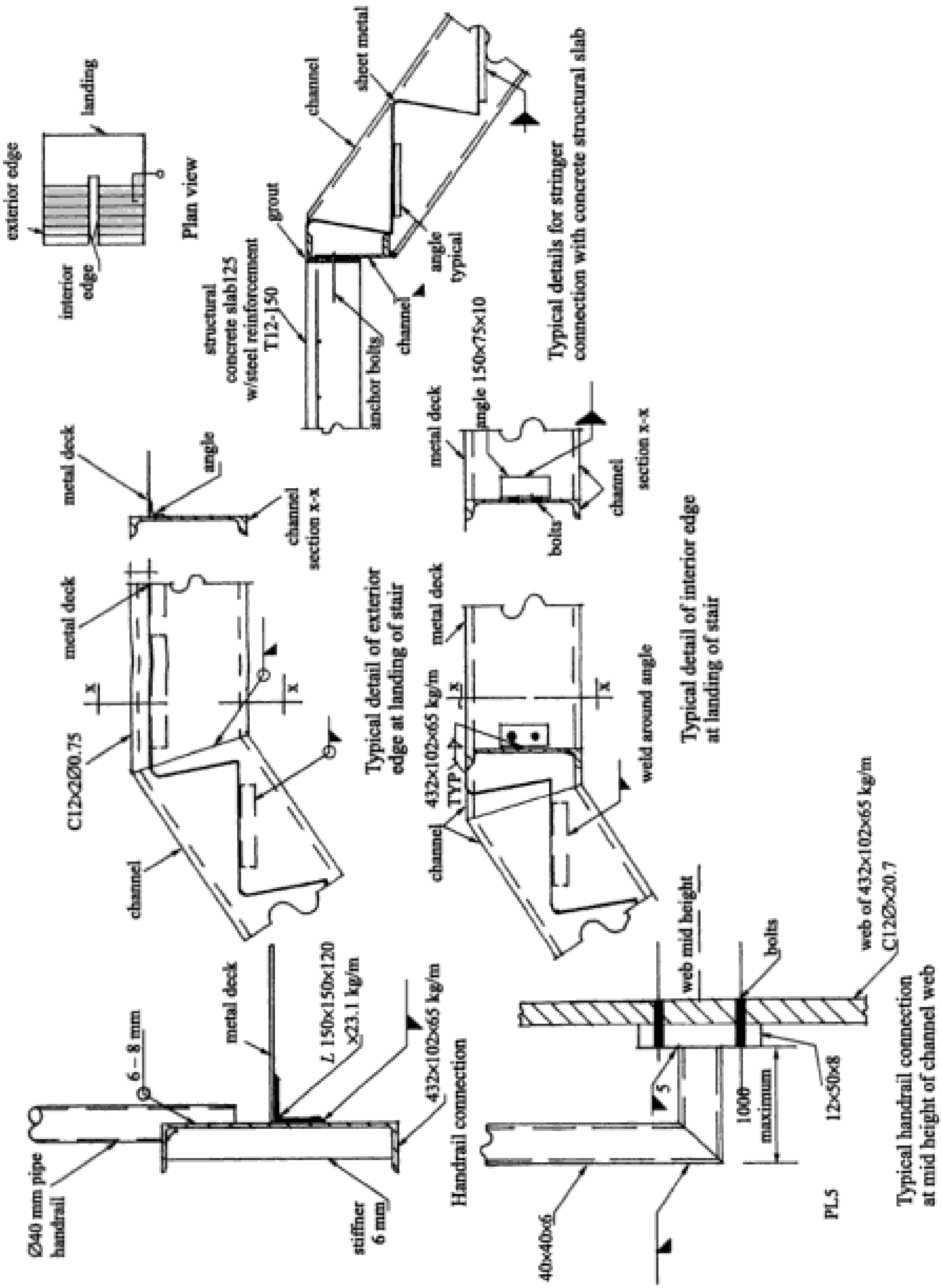

Figure A2.2.7. Connection details for steel stringers to concrete landings and handrails for steel staircases.

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References

DIRECTLY RELATED PUBLICATIONS

Atrops H. (1966). *Pruffahige Statische Hochbauberchnungen in Zahlenbeispielen*, Teil II. Werner-Verlag, Dusseldorf.

Bangash M. Y. H. (1989). *Concrete and Concrete Structures – Numerical Modelling and Applications*. Elsevier, London, 680 pp.

Bangash M. Y. H. (1993). *Impact and Explosion – Analysis and Design*. Blackwell Scientific Publications, Oxford, 850 pp.

Bangash M. Y. H. (1992). *Structural Detail in Concrete*. Blackwell Scientific Publications, Oxford.

Bergman R. V. (1956). Helicoidal staircases of reinforced concrete. *JACI*, (October): 403-412.

Beryer K. (1956). *Die Statik im Stahlbetonbau, 2. Aufl. Springer-Verlag*. Berlin.

Born J. (1965). In: *Faltwerk. Beitrag S. 385-454, Beton-Kalender*, (1965). II Teil. W. Ernst & Sohn, Berlin/Munchen.

Bourdin A. (1965). Escalier sur limon helicoidal unique calcul du limon. Annales de l'Institut Technique du Baitment et des Travaux, Publics No 205 (Jan.), pp. 80-122.

Brendel G. (1963). *Stahlbetonbau*, 4. Aufl. B G Teubner, Leipzig.

British Standards Institute (1977). Code of practice for stairs, BS5395.

British Standards Institute (1984). Stairs, ladders and walkways – Part 2. Code of practice for the design of helical and spiral stairs, BS5395; (1985). Part 3. Code of practice for the design of industrial type of stairs, permanent ladders and walkways, BS5395.

Chandrashekhara K. and Srinivasan S. P. (1972). Photoelastic analysis of free-standing stairs. *Journal of the Structural Division, ASCE, Proc.*, 98(ST12): 2836-2841. Paper 9400 (Dec.).

Chriss S., Glogau O. A. and Leibenberg A. C. (1966). Discussion of experimental study of a free standing staircase. *American Concrete Institute Journal, Proc.*, 63(12): 1487-1494.

Cohen J. S. (1955). Design of helical staircases. *Concrete and Constructional Engineering (London)*, 50(5): 181-194.

Cusens A. R. (1960). Note on analyses of helical stairs. *Concrete and Constructional Engineering (London)*, 55(12): 457.

Cusens A. R. and Kuang J. G. (1966). Analysis of free standing stairs under symmetrical loading. *Concrete and Constructional Engineering (London)*, 60(5): 167-172.

Cusens A. R. and Kuang J. G. (1966). Experimental study of a free standing staircase. *American Concrete Institute Journal, Proc.*, 63(5): 586-604.

Cusens A. R. and Santathadaporn S. (1966). *Design Charts for Helical Stairs with Fixed Supports*. Concrete Publications Ltd., London, 35 pp.

Cusens A. R. and Trirojna S. (1964). Helical staircase study. *JACI*, (January): 85-101.

Deutscher Beton-Verein E. V. (1974). *Beispiele zur Bemessung nacho in 1045*. Wiesbaden, Berlin.

DIN 4102 (1974). Teil 4 Brandverhalten von Baustoffen und Bauteilen. Einreihung in die Bergriffe Beton-Kalender, I Teil S, 1267-1283.

DIN 1055 (1978). Lastannahmen fur Bauten Teil 1-6 Beton-Kalender, II Teil, S. 31-109. W. Ernst & Sohn.

DIN 1080 (1977). Teil 1 Begriffe, formeizeichen und Einheiten im Bauingenieurwesen: Grundlagen, Beton-Kalender, II Teil. W. Ernst & Sohn.

Ersan T. Kirissiz sanhanlikli merdivenler, Beitrag S 50-71. In: *Turkische Bauingenieurbund Technische Mitteilung*, Heft 1 (in Turkish).

Franz G. (1969). *Konstruiktionsleher des Stahlebtons, Erster Band*, 3 Aufl. Springer-Verlag, Berlin/Heidelberg/New York.

Franz G. (1969). *Konstruiktionslehre des Stahlebtons, Zweiter Band*. Springer-Verlag, Berlin/Heidelberg/New York.

Fuchissteneir W. (1954). Die Freitragende Wendeltreppe. *Beton und Stahlbetonbau*, 49(11): 252-259.

Fuchissteneir W. (1954). Die Freitragende Wendeltreppe. *Beton und Stahlbetonbau*, Berlin, West Germany, 49(11): 252-259.

Fuchissteneir W. (1965). *Treppen Beitrag S 51-84 in Beton-Dalender, II Teil*. W. Ernst & Sohn, Berlin/Munchen.

Gerstle K. H. and Clough R. W. (1955). Model analyses of three-dimensional slab structures. *Proc. Society for Experimental Analysis*, 13(2).

Gould P. L. (1963). Analysis and Design of a Cantilever Staircase. *American Concrete Institute Journal, Proc.*, 60(7): 881-899.

Grasser E. (1978). *Bemessung für Biegung mit Längskraft, Schub und Torsion. Beitrag S. 680-812 in Beton-Kalender*, I. Teil. W. Ernst & Sohn, Berlin/Munchen/Dusseldorf.

Guerrin A. (1971). *Traite de Beton Arme*, Tome, IV, 4, Aufl. Dunod, Paris.

Hajanl-Konyi K. (1959). Test of a staircase. *Concrete and Constructional Engineering (London)*, 54(1): 25-27.

Holmes A. M. C. (1957). Analysis of helical beams under symmetrical loading. *Journal of the Structural Division, ASCE, Proc.*, 83(ST6): 1-37. Paper 1437.

Howe J. R. (1949). *Simplified Stair Building*. Crosbt Lockwood, London.

Kersten C. und Kunhnert H. (1957). *Der Stahlbetonbau*, 10 Aufl.

Leonhardt F. (1977). *Voriesungen uber Massivbau*, Vierter Teil. Springer-Verlag, Berlin/Heidelberg/New York.

Leonhardt F. und Monnig E. (1977). *Voriesungen uber Massivbau*, Dritter Teil, 3. Aufl. Springer-Verlag, Berlin/Heidelberg/New York.

Leonhardt F. und Monnig E. (1973). *Voriesungen uber Massivbau*, Erster Teil, 2. Aufl. Springer-Verlag, Berlin/Heidelberg/New York.

Leonhardt F. und Monnig E. (1975). *Voriesungen uber Massivbau*, Zweiter Teil, 2. Aufl. Springer-Verlag, Berlin/Heidelberg/New York.

Liebenberg A. C. (1962). Discussion of analysis of free straight multiflight staircases, by A. Siev. *Journal of the Structural Division, ASCE, Proc.*, 88(ST6): 327-333. Paper 3376.

Liebenberg A. C. (1960). The design of slab type reinforced concrete stairways. *The Structural Engineer (London)*, England, 38(5): 156-164.

Mattock A. H. (1957). Design and construction of a helical staircase. *Concrete and Constructional Engineering (London)*, 52(3): 99-105.

Maunsur M. A. and Rangan B. V. (1978). Torsion in spandrel beams. *Journal of the Structural Division ASCE, Proc.*, 104(ST7): 1061-1075. Paper 13861.

Mitchell L. H. and Shaw F. S. (1962). Column-less stairs. *Architectural Science Review*, Sydney, Australia, 5(2): 80.

Morgan V. A. (1960). Comparison of analyses of helical stairs. *Concrete and Constructional Engineering (London)*, 55(3): 127-132.

Ng S. F. and Chetty A. T. (1975). Study of three flight free-standing staircase. *Journal of the Structural Division, ASCE, Proc.*, 101(ST7): 1419-1434. Paper 11434.

NHBC (1991). Standards Vols. 1 and 2. Amersham, Buckinghamshire. National House Building Council, UK.

Ozmen G. (1960). Elastik mesnetli donel merdivenier. Technische Universitat Istanbul-Zeitschrift Heft, 18/1 (in Turkish).

Rahlwes K. (1977). *Lagerung und Lager von Bauwerken, Keitrag S, 981-1084 in Beton-Kalender*, II Teil. W. Ernst & Sohn, Berlin/Munchen/Dusseldorf.

Rajogopalan K. S. (1973). Design of a free-standing staircase. *American Concrete Institute Journal, Proc.*, 70(10): 714-716.

Ryder D. E. (1961). Spiral staircases. *Proc. Institution of Structural Engineers*, (February): 67-70.

Saenz P. L. and Martin I. (1961). Slabless tread – riser stairs. *JACI* (September).

Sari A. (1968). Duzy Sirkulasyon araclari – Merdivenier. Technische Universitat Istanbul (in Turkish).

Sauter F. (1964). Free standing stairs. *American Concrete Institute Journal, Proc.*, 61(7): 847-870.

Schneider K. (1969). Zur Berechnung Freitragender Wendeltreppen mit torsionssteifem Lauftrager fur Verschiedene Auflagebedingungen. *Der Stalbau*, 38(3): 93-96.

Scordelis A. C. (1960a). Closure to discussion of 'Internal Forces in Uniformly Loaded Helicoidal Girders'. *ACI Journal Proc.*, 57(6): 1491-1502.

Scordelis A. C. (1960b). Internal forces in uniformly loaded helicoidal girders. *ACI Journal Proc.*, 56(10): 1013-1026.

Siev A. (1962). Analysis of free straight multiflight staircases. *Journal of the Structural Division, ASCE, Proc.*, 88(ST3): 207-232. Paper 3168.

Siev A. (1963). Closure to 'Analysis of free straight multiflight staircases'. *Journal of the Structural Division, ASCE, Proc.*, 89(ST5): 251-254. Paper 3688.

Solanki T. H. (1976). Helicoidal girders. *Journal Structural Division, ASCE, Proc.* (Technical note ST4): 869-873.

Stiglat K. und Wippel H. (1977). *Massive Platten, Beitrag S. 311-419 in Beton, Kalender*, I Teil. W. Ernst & Sohn, Berlin/Munchen/Dusseldorf.

Taleb E. J. (1964). The analysis of stairs with unsupported intermediate landings. *Concrete and Constructional Engineering* (*London*), 59(9): 315-320.

Trirojna S. (1962). The Design and Behaviour of Fixed-Ended Helicoidal Staircases of Reinforced Concrete, Thesis No. 25. SEATO Graduate School of Engineering, Bangkok, 94 pp.

Uzden K. (1971). Beonarme I, Istanbul. Verlag Technische Universitat (in Turkish).

Walter H. (1958). Zur Abstutzung Freitragender Wendeltreppen Beltrag S. *Beton und Stahlbetonbau*, Vol. 53: 64-68.

Worch G. (1968). *Elastische Schalen, Beitrag S. 263-289, Beton Kalender*, II Teil. W. Ernst & Sohn, Berlin/Munchen/Dusseldorf.

Young Y. F. and Scordelis A. C. An analytical and experimental study of helicoidal girders. *ASCE*, 84(ST5): 29.

SPECIALLY RELATED PUBLICATIONS

ACI (1994). Manual of Concrete Practice. Parts 1 to 5. American Concrete Institute, Detroit, USA.

AISC (1986). Manual of Steel Construction – Load of Resistance Factor Design. American Institute of Steel Construction.

AITC (1991). Timber Construction Manual. American Institute of Timber Construction. Wiley, New York.

Baron F. and Michalos J. P. (1952). Lateral load plane structures and structures curved in space. *Transactions ASCE*, 117: 279-316.

BSI (1985). The Structural use of Steel Work in Buildings, Part 1. British Standards Institute, Milton Keynes.

Corns J. A. (1959). *Staircase Manual*. Cleaver Hume Press Limited, London.

Crawley S. W. and Dillon R. M. (1993). *Steel Buildings – Analysis and Design*, 4th edn. Wiley.

Ihsanullah K. O. (1976). Design Studies of Free-Standing Straight and Helicoidal Staircases. MSC Thesis, University of Dundee.

Magnel G. (1951). Revolutionary staircases built of prestressed concrete. *Civil Engineering*, 21(9): 25-27; with discussions in No. 11, 1951 p. 56; Vol. 22 No. 2, 1952 p. 52 and No. 5, 1952 p. 56.

Michalos J. (1952). Analysis of skewed rigid frames and arches. *ACI Journal, Proc.*, 48(February): 437-454.

Michalos J. (1953). Numerical analysis of frames with curved girders. *Proc. ASCE*, 79(Separate No. 250, August).

Oesterblom I. (1952). Bending and torsion in horizontally curved beams. *ACI Journal*, 48(February): 455-456.

Ozelton E. C. and Baird J. A. (1979). *Timber Designers Manual*. Granada Publications.

Rangan V. (1978). Lower bound limit design of concrete structures. *ASCI*, ST 12(December): 1901-1906.

Solanki I. H. T. (1975). Technical note on free-standing stairs with slabless tread-risers. *Journal of the Structural Division ASCE, Proc.*, 101(ST8): 1733-1738.

Szabo G. (1959). Naeherurgsformeln fuer die Berechnung freitragender Podesttreppen. *Die Bautechnik, Berlin, Germany*, 36(7).

West J. R. (1968). A Theoretical and Experimental Study of Un-Symmetrical Straight Free-Standing Stairs. Honours Thesis, University of Dundee.

RELATED FOREIGN STANDARDS

USA

ANSI (American National Standard Institute) (A10.18) (1983). Construction and Demolition Operation – Temporary Floor and Wall Openings, Flat Roofs, Stairs, Railings and Toeboards – Safety Requirements.

ANSI (A1264.1) (1989). Safety Requirements for Workplace Floor and Wall Openings, Stairs and Railing Systems.

ASAE (American Society of Agriculture Engineers) (S412.1) (1991). Ladders, Cages, Walkways and Stairs.

CSI (05510) (1987). Metal Stairs. (Division 5 – Metals).

CSI (05715) (1990). Prefabricated Spiral Stairs. (Division 5 – Metals).

NAAMM (National Association of Agricultural Metal Manufacturers) (1982). Metal Stairs Manual. Fourth Edition.

FRANCE

Afnor (1986). Basis for Design of structure, Working Loads for Buildings. NF p06-001.

Afnor (1976). Stairs Treads and Risers made from Cement for Straight Flights. NF p87-301.

Afnor (1945). Structural Dimensions, Straight Masonry Stairs. NF p01-011.

Afnor (1993). Wooden Stairs Specifications. NF p21-211.

Afnor (1993). Wooden Stairs Terminology. NF p21-210.

GERMANY

DIN (1971). Design Loads for Buildings – Live loads. DIN 1055, Teil 3.

DIN (1991). Design of Steel Stairs with Open Bar Grading Treads. DIN 24530.

DIN (1991). Open Bar Grating Treads for Steel Stairs. DIN 2453.

DIN (1975). Stage and Studio Set-Up – Steps and Stairs. DIN 15920, Teil 2.

DIN (1978). Stage and Studio Set-Up; Safety Regulations for Practicals, Ramps, Steps, Stairs and Stage Balustrades. DIN 15920, Teil 1.

DIN (1978). Stairs – Terms. DIN 18064.

DIN (1988). Stairs and Ladders; Internal Stairs; Assembly. VG85222, Teil 1.

DIN (1988). Stairs and Ladders; Internal Stairs; Mud Guard. VG85222, Teil 5.

DIN (1988). Stairs and Ladders; Internal Stairs; Strings. VG85222, Teil 2.

DIN (1979). Stairs and Ladders; Ladders; Dimensions, Installation. VG85207.

DIN (1985). Stairs and Ladders; Outside Stairs Assembly. VG85223, Teil 1.

DIN (1985). Stairs and Ladders; Outside Stairs Strings. VG85223, Teil 2.

DIN (1985). Stairs and Ladders; Outside Stairs; Handrails. VG85223, Teil 3.

DIN (1992). Stairs and Ladders; Steps. VG85224.

DIN (1984). Stairs in Building – Main Dimensions. DIN 18065.

DIN (1985). Stairs with Load-Bearing Bolts for Residential Buildings; Design and Execution. DIN 18069.

RELATED REFERENCES ON TORSION

Collins M. P. (1968). Ultimate Strength of Reinforced Concrete Beams Subjected to Combined Torsion and Bending. Torsion of Structural Concrete (SP-18). American Concrete Institute, pp. 379-402. Detroit.

Collins M. P., Walsh P. F. and Hall A. S. (1968). Discussion of ultimate strength of reinforced concrete beams in torsion and shear. *ACI Journal Proc.*, 65(September): 786-788.

Collins M. P. and Lampert P. (1971). Designing for torsion. *Structural Concrete Symposium.* University of Toronto, Civil Engineering Department, Toronto: (May), pp. 38-79.

Collins M. P. and Lambert P. (1973). Redistribution of Moments at Cracking – The Key to Simpler Torsion Design? Analysis of Structural Systems for Torsion (SP-35). American Concrete Institute, Detroit, pp. 343-383.

EuroCode 2 (1992). Design of Concrete Structures. Part DDENV, 1992-1-1.

Good C. D. and Helmy M. A. (1968). Ultimate Strength of Reinforced Concrete Beams in Combined Bending and Torsion. Torsion of Structural Concrete (SP-18). American Concrete Institute, Detroit, pp. 357-377.

Hsu T. T. C. (1968). Torsion of Structural Concrete – A Summary of Pure Torsion. Torsion of Structural Concrete (SP-18). American Concrete Institute, Detroit, pp. 165-178; Also Portland Cement Association Development Department Bulletin D133.

Hsu T. T. C. (1968). Torsion of Structural Concrete – Behaviour of Reinforced Concrete Rectangular Members. Torsion of Structural Concrete (SP-18). American Concrete Institute, Detroit, pp. 261-306; Also Portland Cement Association Development Department Bulletin D135.

Hsu T. T. C. (1968). Torsion of Structural Concrete – Plain Concrete Rectangular Sections. Torsion of Structural Concrete. (SP-18). American Concrete Institute, Detroit, pp. 203-238; Also Portland Cement Association Development Department Bulletin D133.

Hsu T. T. C. (1968). Ultimate torque of reinforced rectangular beams. *Journal of Structural Division, ASCE*, 94(ST2): 485-510; Also Portland Cement Association Development Department Bulletin D127.

Kevin D., Below B., Rangan V. and Stanley H. A. (1975). Theory for concrete beams in torsion and bending. *Journal of Structural Division, ASCE*(August, ST8): 1645-1660.

Lampert P. (1973). Postcracking Stiffness of Reinforced Concrete Beams in Torsion and Bending, Analysis of Structural Systems for Torsion. (SP-35). American Concrete Institute, Detroit, pp. 385-433. Also presented at 1971 Annual Convention, American Concrete Institute, Denver, March, 1971.

Lampert P. (1970). Torsion und Biegung von Stahlbetonpalken. (Torsion and Bending of Reinforced concrete Beams.) Bericht Nr, 27 ETCH Zurich: Institu fur Baustatik (January).

Lampert P. and Thürlimann B. (1971). Ultimate Strength and Design of Reinforced concrete Beams in Torsion and Bending. Publications, International Association for Bridge and Structural Engineering, 31-1, pp. 107-131.

Lessig N. N. (1959). Determination of Load Carrying Capacity of Reinforced Concrete Element with Rectangular Cross-Section Subjected to Flexure with Torsion. Work No. 5, Institute Betona; Zhelezobetona (Concrete and Reinforcement Concrete Institute), Moscow, pp. 5-28; Also available as Foreign Literature Study No. 371, PCA Research and Development Labs, Skokie, Illinois.

Mirza M. S. and McCutcheon J. O. (1968). Discussion of torsion of structural concrete – interaction surface for combined torsion, shear and bending in beams without stirrups. *ACI Journal Proc.*, 65(July): 567-570.

Mirza M. S. and McCutcheon J. O. (1969). Behaviour of reinforced concrete beams under combined bending, shear and torsion. *ACI Journal Proc.*, 66(May): 421-427; Disc. pp. 940-942.

Osburn D. L. et al. (1969). Strength of reinforced concrete beams with web reinforcement in combined torsion, shear and bending. *ACI Journal Proc.*, 66(January): 31-41; Disc. pp. 593-595.

Rausch E. (1929). *Berechnung des Eisenbetons gegen Verdrehung und Abscheren (Design of Reinforced Concrete for Torsion and Shear)*. Springer-Verlag, Berlin.

Victor D. J. and Ferguson P. M. (1968). Beams under distributed load creating moment, shear and torsion. *ACI Journal Proc.*, 65(April): 295-308; Disc. pp. 892-894.

Victor D. J. et al. (1976). Ultimate torque of reinforced concrete beams. *Journal of Structural Division, ASCE*, 102(July, ST7): 1337-1352.

Wang C. (1970). *Matrix Methods of Structural Analysis*, 2nd edn, Ch. 12 and 15. American Publishing Co., Madison, Wisconsin.

Zia P. (1968). Torsion Theories for Concrete Members. Torsion of Structural Concrete (SP-18). American Concrete Institute, Detroit, pp. 103-132.

Zia P. (1970). What do we know about torsion in concrete members? *Journal of Structural Division, ASCE*, 9(June, ST6): 1185-1199.

Hidden page

In recent years both free-standing and geometric staircases have become quite popular. Many variations exist, such as spiral, helical, and elliptical staircases, and combinations of these. A number of researchers have come forward with different concepts in the fields of analytical and numerical design and of experimental methods and assessments. The aim of this book is to cover all these methods and to present them with greater simplicity to practising engineers.

Staircases is divided into five chapters: Specifications and basic data on staircases; Structural analysis of staircases – Classical methods; Structural analysis of staircases – Modern methods; Staircases and their analyses – A comparative study; Design analysis and structural detailing. Charts and graphs are included and numerous design examples are given of free-standing and other geometric staircases and of their elements and components. These examples are related to the case studies which were based on staircases that have already been constructed. All examples are checked using various Eurocodes.

The book includes bibliographical references and is supported by two appendices, which will be of particular interest to those practising engineers who wish to make a comparative study of the different practices and code requirements used by various countries; detailed drawings are included from the USA, Britain, Europe and Asia. *Staircases* will serve as a useful text for teachers preparing design syllabi for undergraduate and post graduate courses. Each major section contains a full explanation which allows the book to be used by students and practising engineers, particularly those facing the formidable task of having to design/detail complicated staircases with unusual boundary conditions. Contractors will also find this book useful in the preparation of construction drawings and manufacturers will be interested in the guidance given in the text.

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